September 9, 2015 4 p.m. – S33

Perron theorem and Coxeter groups

Comprehensible Seminar

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Perron's theorem says that the spectral radius of a positive matrix is a simple eigenvalue strictly greater than the modulus of the other eigenvalues. In the classical geometric representation of Coxeter groups, matrices are never positive but their spectral properties seem to be like positive matrices. In this talk, we show a criterion to check if the spectral radius of a real matrix (corresponding to an element of a Coxeter group) is a simple strictly dominant eigenvalue. If time allows, we will present open problems in the study of fractals and Coxeter groups that motivate this work.

This is a joint work with Jean-Philippe Labbé.

Groupes de Coxeter et théorème de Perron
séminaire compréhensible!

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9 sept 2015

Reflection Group

V: n-dim. vector space

$(\cdot,\cdot)$: symmetric bilinear form

If $(\cdot,\cdot)$ is pos. definite, then $\langle V, (\cdot,\cdot)\rangle$ is an Euclidean space.
↳ inner product.

Let $\alpha \in V$. A *reflection* is a linear map that sends $\alpha \mapsto -\alpha$ and fixes the hyperplane $H_\alpha = \{\lambda \in V \mid (\lambda,\alpha)=0\}$ orthog. to α.

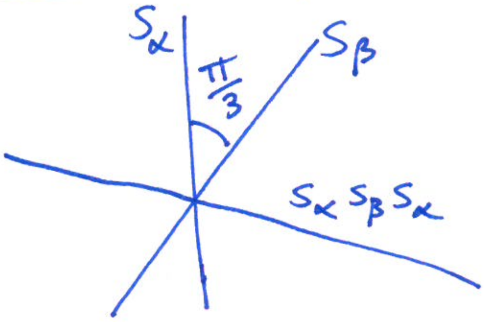
formula: $S_\alpha(\lambda) = \lambda - 2\dfrac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$

Orthogonal transformations are $O(V) = \{\text{lin. map } f \mid (f(\lambda), f(\mu)) = (\lambda,\mu)\}$
the set of lin map that preserve the bilinear form

A *reflection group* is a subgroup of $O(V)$ generated by reflections.

Question Classify all Reflection Groups ~~[illegible]~~.

EX:



Rotations: $S_\alpha S_\beta$, $S_\beta S_\alpha$

Identity: e

Ref

Humphreys 92
Björner, Brenti, 2005
Borovik, Borovik, 2009
Armstrong, 2009

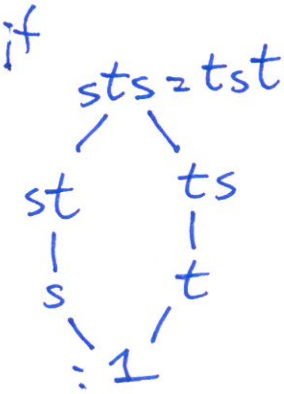
Groupe de Coxeter is a group withe the presentation

$$\langle r_1, r_2, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

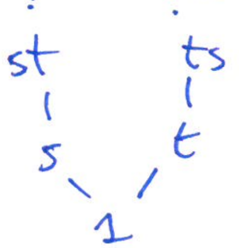
where $m_{ii} = 1$ and $m_{ij} \in \{2,3,4,\ldots,\infty\}$ if

Notes
- $m_{ii} = 1 \Rightarrow r_i$ is an involution
- $m_{ij} = 2 \Rightarrow r_i$ and r_j commute
- $m_{ij} = m_{ji}$ $\forall i,j$.

$\boxed{EX}$ $\langle s,t \mid s^2 = t^2 = (st)^3 = 1\rangle$



$\langle s,t \mid s^2 = t^2 = 1\rangle$
Infinite dihedral group



Notation W = group, $S = \{r_1, \ldots, r_n\}$ generators

Thm (Coxeter, 1934) Every reflection group is a Coxeter Group.

Thm (Coxeter, 1935) Every finite Coxeter group has a representation as a reflection group.

He also gave a classification of finite Coxeter groups according to the matrix $(m_{st})_{st \in S \times S}$ using Coxeter graphs.

"has a representation": S: ens. de generators

V = vector space generated by vectors $\{\alpha_s \mid s \in S\} = \Delta$

symmetric bilinear form:

$$B(\alpha_s, \alpha_t) := \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) \\ -1, & \text{si } m_{st} = \infty \end{cases}$$

$\forall s \in S$, define the reflection

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s$$

The representation is:

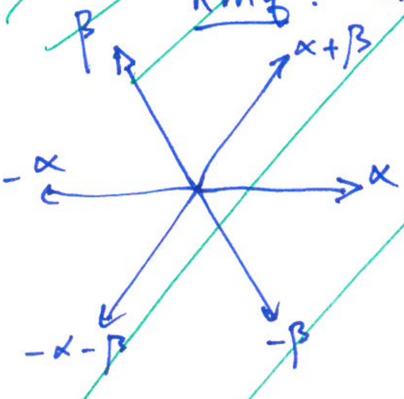
$$\rho : W \hookrightarrow GL(V)$$
$$s \longmapsto \sigma_s$$

Since ρ is injective, σ_s preserves B, then W isomorphic to a subgroup of $O(V)$ generated by reflections. It is a reflection group.

TOO LONG! TOO ABSTRACT EXAMPLE IS BETTER ~~PLEASE~~

Root system

Rmq: $\#\{w \in W \mid w^2 = 1\} \geq \#S$



Def A root system Φ is a finite set of nonzero vectors of V s.t.

(1) $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$, $\forall \alpha \in \Phi$

(2) $s_\alpha \Phi = \Phi$, $\forall \alpha \in \Phi$

Elements of Φ are called roots

(3) $\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$ (crystallographic)

Lemme (Humphreys 1992 — Ref book) Let G be a reflection group. $\exists$ root system Φ s.t. reflections of G are precisely $\{s_\alpha \mid \alpha \in \Phi\}$, i.e. reflections through hyperplanes perpendicular to the roots.

Lemme $\exists$ set Δ of simple roots s.t. $\Phi = W(\Delta)$

and $\exists$ set Φ^+ of positive roots s.t. $\Phi = \Phi^+ \sqcup -\Phi^+$

and $\Delta \subset \Phi^+ \subset \text{cone}(\Delta)$.

Also Δ is a base of V.

SOURCE: "Combinatorial approach to clusters using sortable elements of Coxeter groups" Mémoire de maîtrise, J P Labbé, UQAM, 2010 p.25-26.

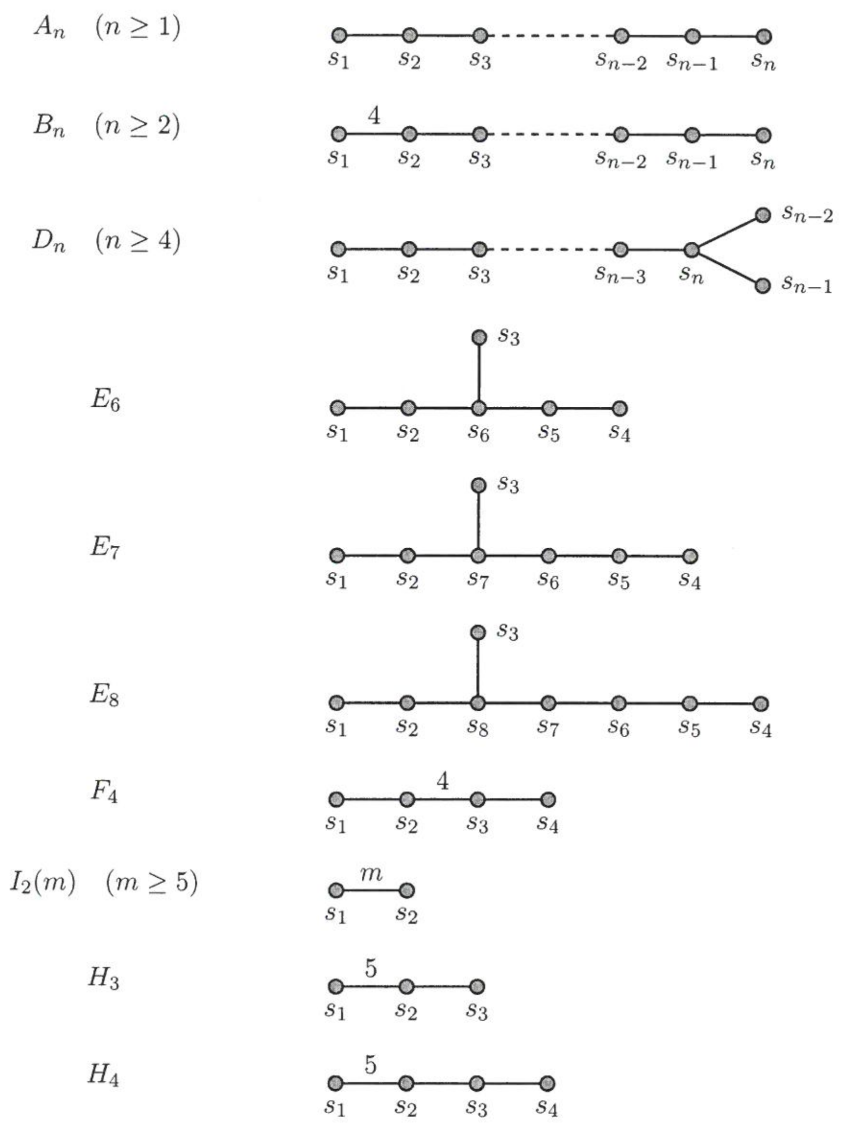


Figure 1.11 Graphe de Coxeter des groupes de Coxeter irréductibles finis. Les groupes de types A,B et D sont souvent appelés les familles infinies et les autres sont appelés groupes exceptionnels.

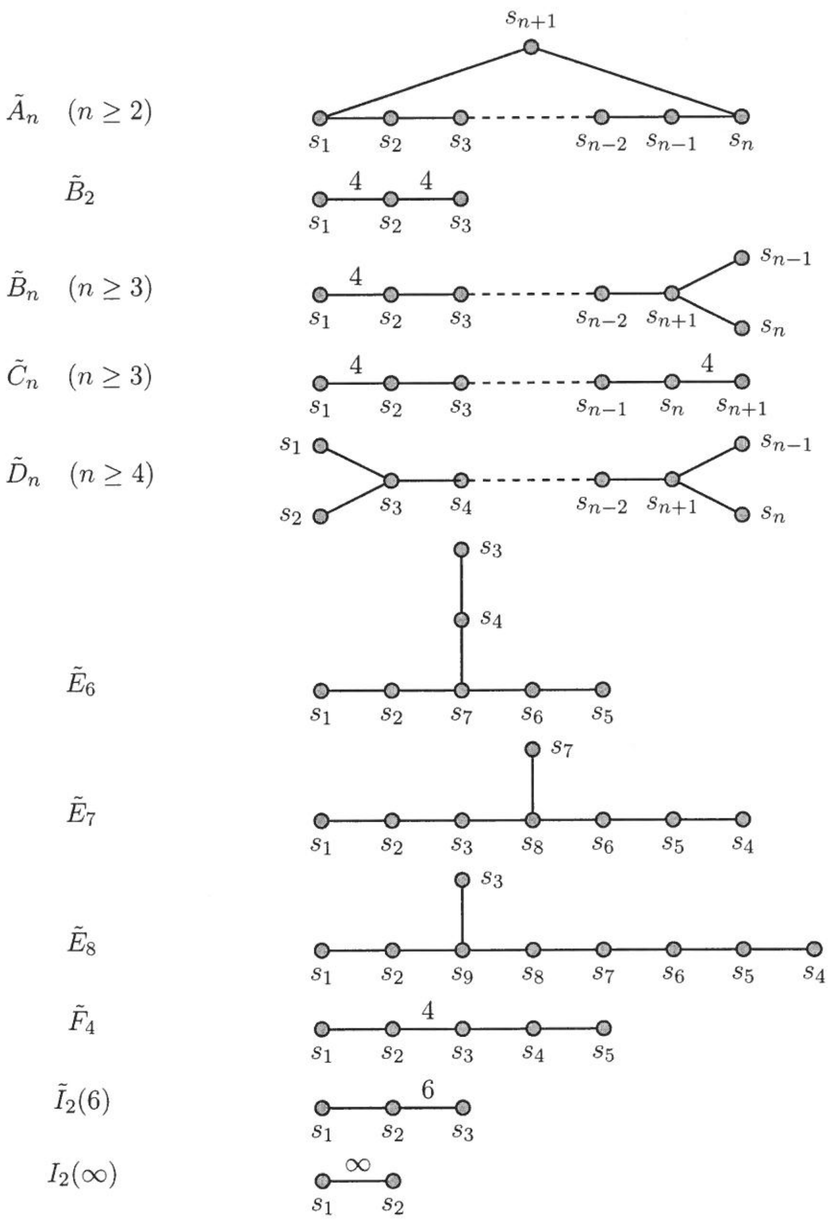


Figure 1.12 Graphe de Coxeter des groupes de Coxeter irréductibles affines.

$\boxed{\text{EX}}$ $$M = (m_{st}) = \begin{pmatrix} m_{ss} & m_{st} \\ m_{ts} & m_{tt} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

V = v.s. generated by the base $= \Delta = \{\alpha_s, \alpha_t\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

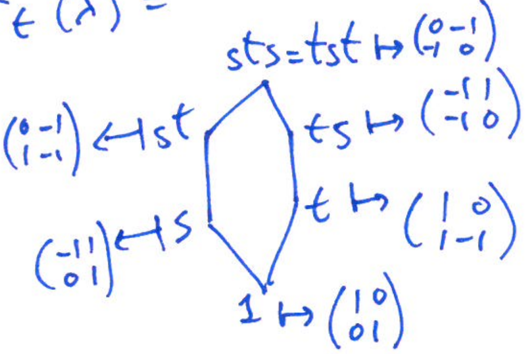
$$B(\alpha_s, \alpha_s) = -\cos\left(\frac{\pi}{1}\right) = 1 = B(\alpha_t, \alpha_t)$$

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2} = B(\alpha_t, \alpha_s)$$

$$B(x, y) = (x_1\ x_2) \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s = [\dots] = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\sigma_t(\lambda) = [\dots] = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

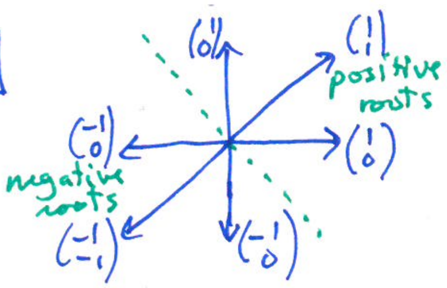


$\boxed{\text{EX}}$ $$M = \begin{pmatrix} m_{ss} & m_{st} & m_{su} \\ & m_{tt} & m_{tu} \\ & & m_{uu} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \\ 3 & 7 & 1 \end{pmatrix} \longrightarrow \text{Infinite Coxeter Groupe on generators } S = \{s, t, u\}$$

We get $$p(sutsutus) = \begin{pmatrix} -4{,}04\dots & 1 & 2{,}24\dots \\ -4{,}04\dots & 1{,}80\dots & 1{,}80\dots \\ -6{,}29\dots & 3{,}24\dots & 2{,}24\dots \end{pmatrix}$$

Remark: The set of columns Φ is stable under W: $W(\Phi) = \Phi$.
In fact $\Phi = W(\Delta)$.

$\boxed{\text{EX}}$



Φ is a root system, fundamental in the study of Coxeter groups.

$\boxed{\text{column} = \text{root}}$

Lemma Let G be a reflection group. $\exists$ root system Φ s.t. reflections of G are precisely the reflections through hyperplanes perpendicular to the roots $\{s_\alpha \mid \alpha \in \Phi\}$.

Positive roots: $\Phi^+ = \text{cone}(\Delta) \cap \Phi$ Negative roots: $\Phi^- = -\Phi^+$.

Partition $\Phi = \Phi^+ \cup \Phi^-$.

Therefore the nonzero entries of a column all have the same sign.

If $|W| = \infty$, then $|\Phi| = \infty$.

Def Limit roots are the accumulation points of roots in the projective space $\mathbb{P}V$.

Thm (Hohlweg, JPLabbé, Ripoll, 2014) Limit roots are on the isotropic cone $\{x \in V \mid B(x,x) = 0\}$.

Conjecture limit roots $\cong$ infinite reduced words$/\sim$

Subcase: Periodic infinite reduced words.

Question $w \in W$ Is it true that columns of powers of $\rho(w)$ all have the same limit in $\mathbb{P}V$?

In other words, is $\rho(w)$ like a positive matrix for which Perron thm applies?

Thm (Perron 1907) $A \in \mathbb{R}^{n\times n}$ primitive, spectral radius λ. Then λ is a (1) simple (2) root of the char. poly (3) strictly greater than the modulus of any other eigenvalue and (4) λ has strictly positive eigenvectors u and v. where $uv = (1)$.

Also $\lim_{k\to\infty} \frac{1}{\lambda^k} A^k = vu$ is of rank one.

Can not be generalized to matrices and their powers with pos. and negative columns while keeping conclusions (1), (2) and (3):

$\boxed{\text{EX}}$ $A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 1 \\ -3 & 1 & 3 \end{pmatrix}$ and $\chi_A(\lambda) = (\lambda-2)^2(\lambda-1)$

But we have this criteria:

Thm (Labbé-2) Let $A \in \mathbb{R}^{n\times n}$ s.t. $Av = \lambda v$, $uA = \lambda u$, $v > 0$, $1^T v = (1)$ and $uv = (1)$. the fol. cond. are equivalent:

(i) (1), (2) and (3)

(ii) $\lim_{k\to\infty} \frac{1}{\lambda^k} A^k = vu$

(iv) $\lambda v 1^T + (I - v1^T)A$ is eventually positive.

$\boxed{\text{EX}}$ $B = \begin{pmatrix} -11 & 14 \\ -24 & 29 \end{pmatrix}$, $v = \frac{1}{20}\begin{pmatrix} 7 \\ 13 \end{pmatrix}$, $u = \frac{1}{6}(-20, 20)$, $\lambda = 15$.

We have $Bv = \lambda v$, $uB = \lambda u$, $1^T v = (1)$ and $uv = (1)$.

we compute
$$\lambda v 1^T + B - v 1^T B = 15 \cdot \frac{1}{20}\begin{pmatrix} 7 & 7 \\ 13 & 13 \end{pmatrix} + B - \frac{1}{20}\begin{pmatrix} 7 & 7 \\ 13 & 13 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 36 & 21 \\ 39 & 54 \end{pmatrix} > 0$$

$\Rightarrow$ 15 is simple eig. v. of B strictly dominating.

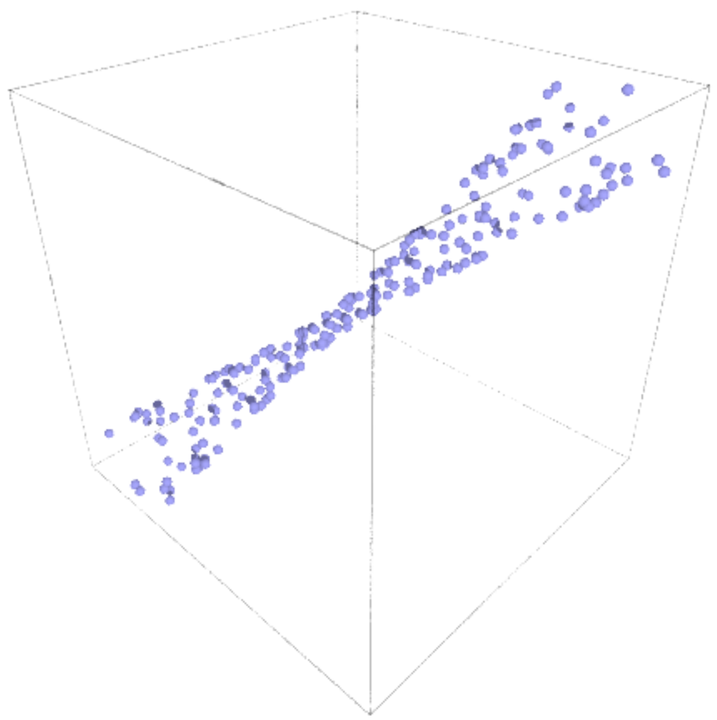
In [40]:
```
def roots(G, depth):
    L = []
    for i in range(depth):
        for w in G.elements_of_length(i):
            for column in w.canonical_matrix().columns():
                column = tuple(entry.real() for entry in column.n())
                L.append(column)
    return L

def draw_roots(G, depth):
    R = roots(G, depth)
    P = points(R)
    if len(R[0]) == 2:
        P.show()
    elif len(R[0]) == 3:
        P.show(viewer='tachyon')
    else:
        raise NotImplementedError
```

In [22]:
```
G = CoxeterGroup([[1,2,3],[2,1,7],[3,7,1]])
G
```

Out[22]:
```
Coxeter group over Universal Cyclotomic Field with Coxeter matrix:
[1 2 3]
[2 1 7]
[3 7 1]
```

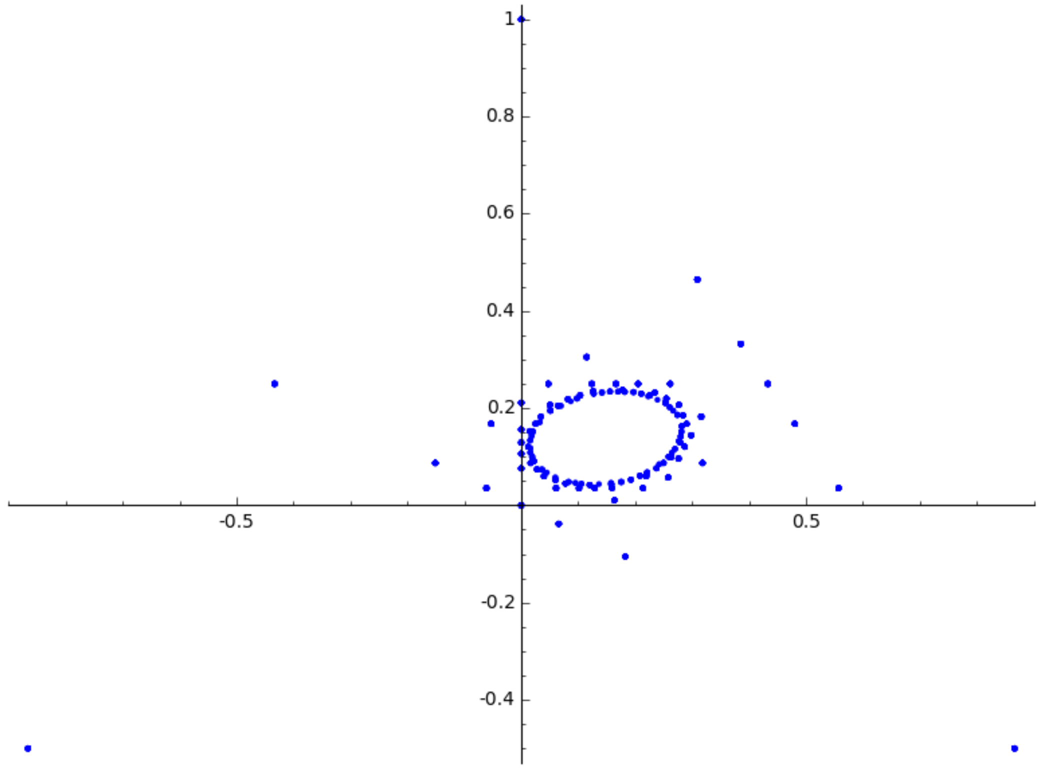
In [23]:
```
draw_roots(G, 15)
```



In [41]:
```
def normalized_roots(G, depth):
    R = roots(G, depth)
    L = [vector(root)/sum(root) for root in R]
    return L

sqrt2 = sqrt(2)
sqrt3 = sqrt(3)
M3to2 = matrix(2,[-sqrt3,sqrt3,0,-1,-1,2],ring=RR)/2
M4to3 = matrix([(1,  0, -1/sqrt2),
                (-1, 0, -1/sqrt2),
                (0,  1,  1/sqrt2),
                (0, -1,  1/sqrt2)], ring=RR).transpose()
def draw_normalized_roots(G, depth):
    R = normalized_roots(G, depth)
    if len(R[0]) == 3:
        P = points([M3to2*root for root in R])
        P.show(frame=False)
    elif len(R[0]) == 4:
        P = points([M4to3*root for root in R])
        P.show(viewer='tachyon')
    else:
        raise NotImplementedError
```

In [39]:
```
draw_normalized_roots(G, 15)
```



SOURCE: "Polyhedral Combinatorics of Coxeter groups" PhD thesis, JP Labbé, Berlin, 2013. p.89-90.

Appendix A

Some root systems of rank 3 & 4

Here are some images of normalized root systems of rank 3 and 4 with small labels.

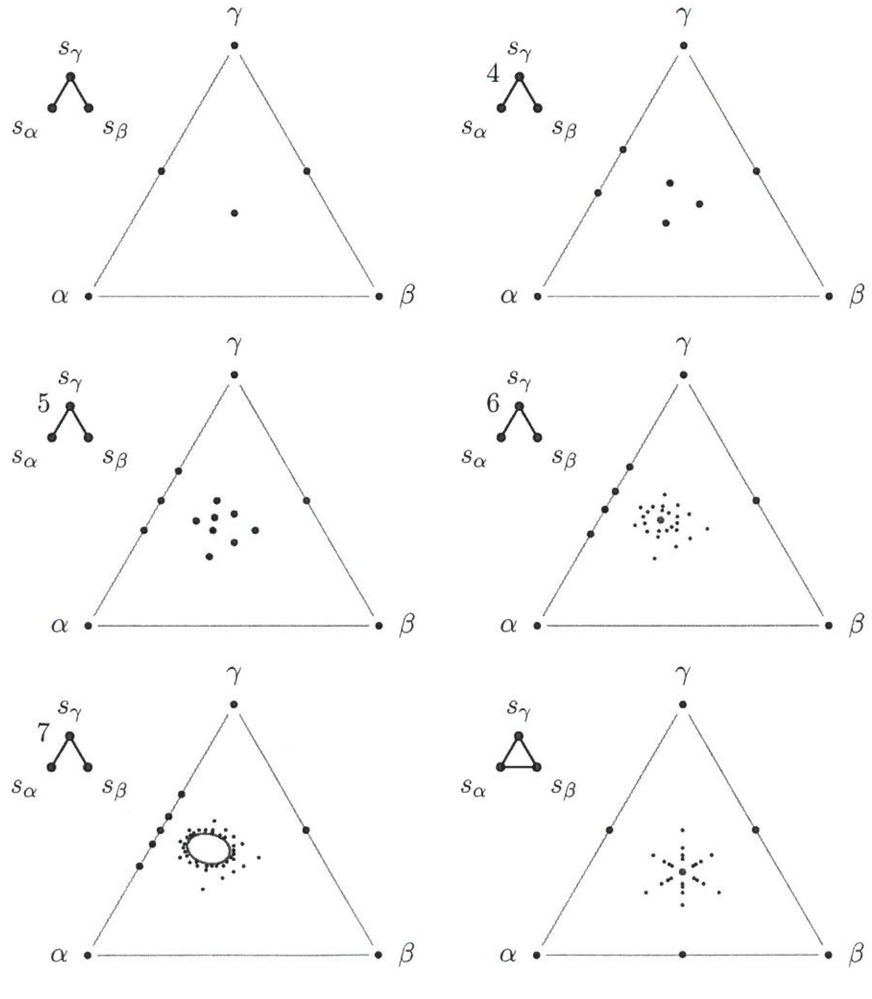


FIGURE A.1: In the first column: type A_3, H_3 and the triangle group $\{2, 3, 7\}$. In the second column: B_3, $\widetilde{I}_2(6)$ and type $\widetilde{A}_2$.

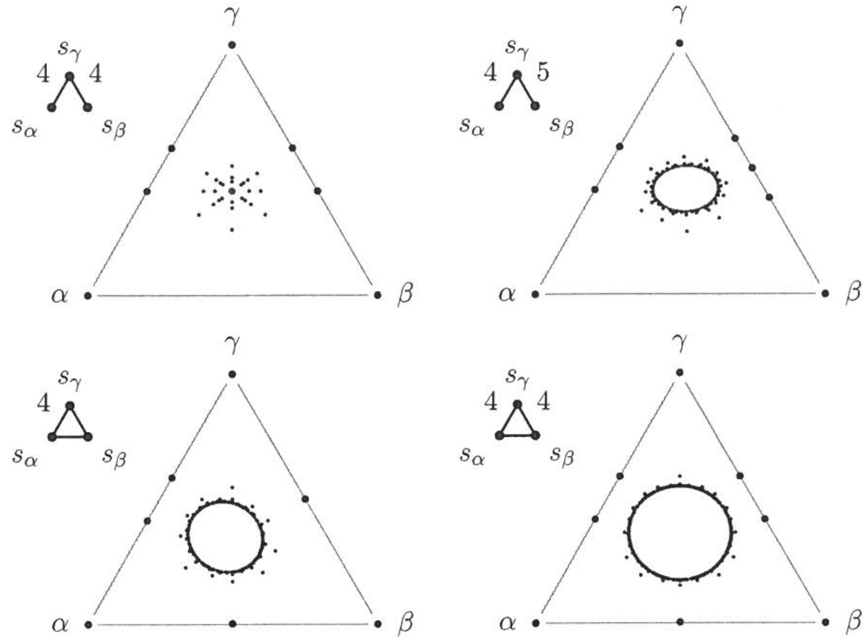


FIGURE A.2: In the first column: type $\widetilde{B}_2$ and the triangle group $\{3, 3, 4\}$. In the second column: the triangle group $\{2, 4, 5\}$ and the triangle group $\{3, 4, 4\}$.

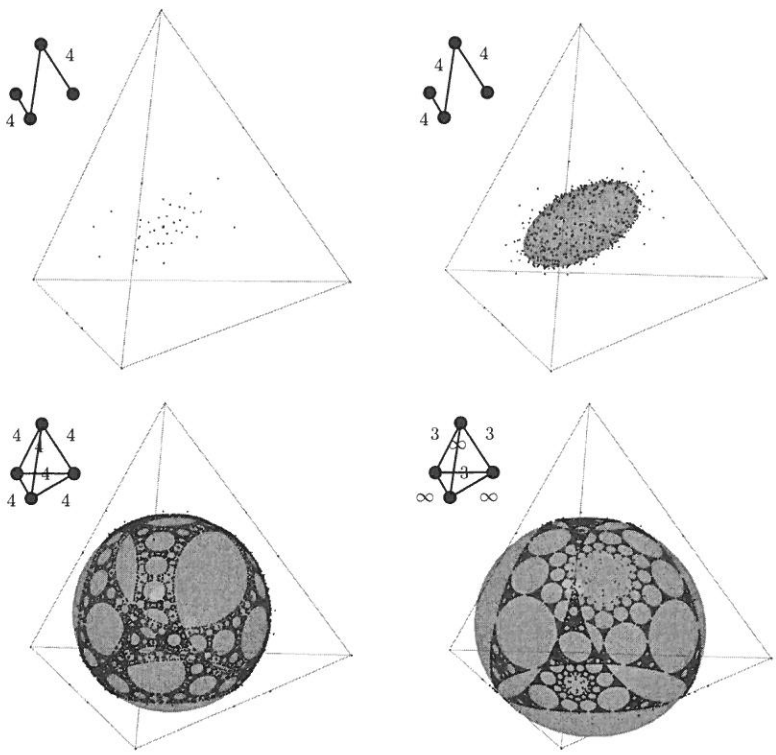


FIGURE A.3: In the top right image: type $\widetilde{C}_3$ and three different groups two of which give rise to fractal limits.