On the dynamics and combinatorics of Jeandel-Rao tilings

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Journées de combinatoire de Bordeaux
13 février 2018
LaBRI, Bordeaux, France

17th Mons Theoretical Computer Science Days

Journées montoises : http://jm2018.sciencecon f.org/

September 10-14, 2018 at LaBRI

SCOPE :

- **word combinatorics** and formal languages from their **different perspectives** (combinatorial, algorithmic, dynamical, logic, ...).
- **welcomes other branches** of computer science and mathematics linked to it (number theory, computability, model checking, semigroups, game theory, discrete geometry, decentralized algorithms, bioinformatics, ...).

SUBMISSION :

- Abstracts between **1 and 4 pages**

IMPORTANT DATES :

- submission before **May 28th 2018**

Outline

Paris

Bordeaux

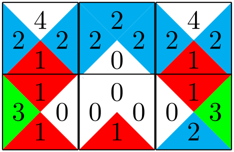
Wang tiles

A **Wang tile** is a square tile with a color on each border

Tile set T : a finite collection of such tiles.
A tiling of the plane : an assignment

$$\mathbb{Z}^2 \to T$$

of tiles on infinite square lattice so that the contiguous edges of adjacent tiles have the same color.



Note : rotation not allowed.

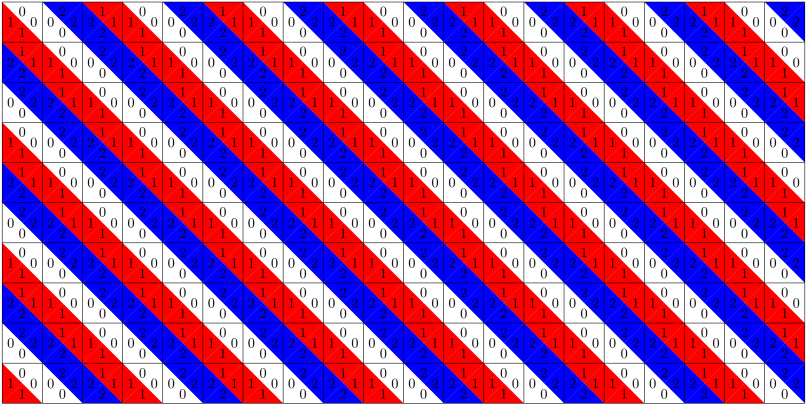
Eternity II puzzle (2007)

- A puzzle which involves **placing 256 square puzzle** pieces into a 16 by 16 grid constrained by the requirement to **match adjacent edges**
- A **2 million prize** was offered for the first complete solution
- **No solution found** before the competition ended on 31 Dec 2010
- At most $256! \times 4^{256} \approx 1.15 \times 10^{661}$ possibilities to check.

Source : `https://en.wikipedia.org/wiki/Eternity_II_puzzle`

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.



A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

Aperiodicity

A tile set is **finite** if there is no tiling of the plane with this set.
A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

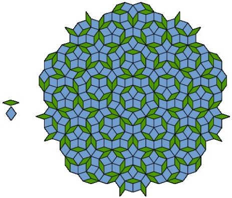
Every set of Wang tiles is either **finite or periodic**.

- 1966 (Berger) : There **exists an aperiodic** set of Wang tiles
- 1976 (Penrose) : discovered an aperiodic set of **two tiles**
- 1982 (Shechtman) : observed that certain aluminium-manganese alloys produced a **quasicrystals structure**
- 2011 : Dan Shechtman receives **Nobel Prize** in Chemistry

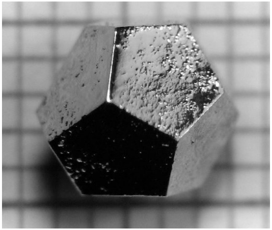
"*His discovery of quasicrystals revealed a **new principle for packing of atoms and molecules***", stated the Nobel Committee that "*led to a **paradigm shift** within chemistry*".

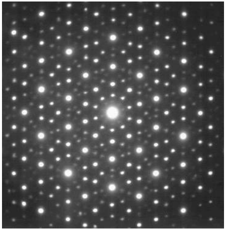
Quasicrystals

Penrose tiles and tiling :



A Ho-Mg-Zn **icosahedral quasicrystal** formed as a pentagonal dodecahedron and its **electron diffraction** pattern :





Source : `https://en.wikipedia.org/wiki/Quasicrystal`

Some notions and results (< 2000)

Definition

A discrete set X in $\mathbb{R}^d$ is a **Delone set** if it is uniformly discrete and relatively dense. It is called a **Meyer set** if the self-difference set $X - X$ a Delone set.

"The notion of Delone sets as fundamental objects of study in crystallography was introduced by the Russian school in the 1930's ; in particular, by Boris Delone [...]. One can think about a Delone set as an idealized model of an ***atomic structure*** *of a material [...]"*

Source : Boris Solomyak, `arxiv:1802.02370`

Theorem (Lagarias, Meyer)

Let X be a Meyer set in $\mathbb{R}^d$ such that $\eta X \subseteq X$ for a real number $\eta > 1$, then η is a **Pisot number** or a **Salem number**.

Other results relates them to **cut-and-project set**, etc.

Regular tetrahedron packing arrangement (2009)

Letter

Disordered, quasicrystalline and crystalline phases of densely packed tetrahedra

Amir Haji-Akbari, Michael Engel, Aaron S. Keys, Xiaoyu Zheng, Rolfe G. Petschek, Peter Palffy-Muhoray & Sharon C. Glotzer

"One of the simplest shapes for which the densest packing arrangement remains unresolved is the ***regular tetrahedron*** *[...]. Using a novel approach involving thermodynamic computer simulations that allow the system to evolve naturally* ***towards high-density states****, Sharon Glotzer and colleagues have worked out the* ***densest ordered packing yet for tetrahedra****, a configuration with a packing fraction of 0.8324. Unexpectedly, the structure is a* ***dodecagonal quasicrystal****, [...] "*

Source (2009) : `doi:10.1038/nature08641` , `https://www.quantamagazine.org/digital-alchemist-sharon-glotzer-seeks-rules-of-emergence-20170`

The Russian meteorite (2016)

*"They **loaded the minerals** found in the rock into a chamber, and then, using a four-meter-long propellant gun, **fired a projectile** into the stack of ingredients. [...] The findings [...] indicate that the **quasicrystals** in the Russian meteorite did indeed **form during a shock event**."*

Source (2016) : `https://www.quantamagazine.org/a-quasicrystals-shocking-origin-20160708/`

Discoveries of aperiodic Wang tile sets (< 1990)

Image credit : http://chippewa.canalblog.com/archives/2010/06/04/18115718.html

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- ...

Ammann 16 tiles

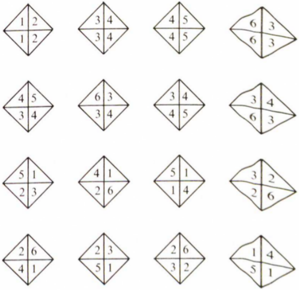


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12. These form the smallest known aperiodic set.

11.1 APERIODIC SETS OF WANG TILES

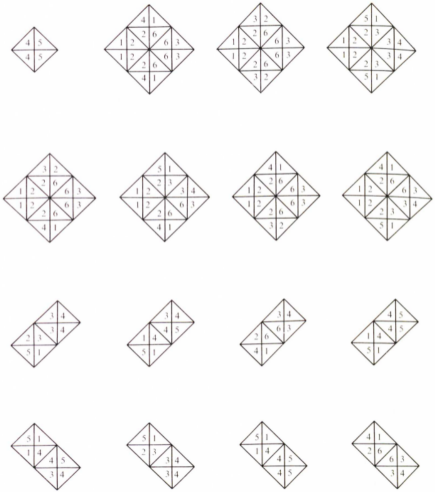


Figure 11.1.16
This diagram shows how the Wang tiles of Figure 11.1.13 can be "decomposed".

11.1 APERIODIC SETS OF WANG TILES 597

Grünbaum, Shephard, Tilings and patterns, 1987, p. 595-597.

Self-similar tilings

Informally, two conditions that imply aperiodicity :

Ammann, Grünbaum, Shephard, 1992

Let $\mathcal{T}$ be a tile set. If

(a) in every tiling admitted by $\mathcal{T}$ there is a **unique way** in which the tiles can be grouped into patches which lead to a tiling by **supertiles** ; and

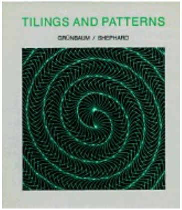
(b) the markings on the supertiles, inherited from the original tiles, imply a matching condition for the supertiles which is **exactly equivalent** to that originally specified for the tiles,

then $\mathcal{T}$ is **aperiodic**.

Mossé 1992 (on $\mathbb{Z}$) ; Solomyak 1998 (in $\mathbb{R}^d$)

A self-similar tiling has the **unique composition property** if and only if it is **nonperiodic**.

Grünbaum, Shephard, Tilings and patterns, 1987



The reduction in the number of Wang tiles in an aperiodic set from over 20,000 to 16 has been a notable achievement. Perhaps the minimum possible number has now been reached. If, however, further reductions are possible then it seems certain that new ideas and methods will be required. The discovery of such remains one of the outstanding challenges in this field of mathematics. One can, of course, look at the problem from the opposite point of view. Is it possible to prove that, for example, 15 tiles are not enough? It is difficult to see how any such proof could be constructed, and the only result we know in this direction is an unpublished theorem of Robinson that no aperiodic set of *four* Wang tiles can exist.

Source : Grünbaum, Shephard, Tilings and patterns, 1987, p. 596.

Discoveries of aperiodic Wang tile sets (< 2000)

Image credit : http://chippewa.canalblog.com/archives/2010/06/04/18115718.html

- 1966 (Berger) : 20426 tiles (lowered down later to 104)
- 1968 (Knuth) : 92 tiles
- 1971 (Robinson) : 56 tiles
- 1971 (Ammann) : 16 tiles
- 1987 (Grunbaum) : 24 tiles
- 1996 (Kari) : 14 tiles
- 1996 (Culik) : (same method) 13 tiles

ELSEVIER

Discrete Mathematics 160 (1996) 259–264

DISCRETE MATHEMATICS

Note

A small aperiodic set of Wang tiles

Jarkko Kari*

Department of Computer Science, University of Iowa, MacLean Hall, Iowa City, Iowa 52242-1419, USA

Received 3 January 1995

Abstract

A new aperiodic tile set containing only 14 Wang tiles is presented. The construction is based on Mealy machines that multiply Beatty sequences of real numbers by rational constants.

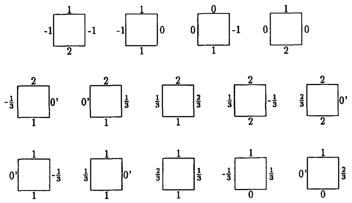


Fig. 1. Aperiodic set of 14 Wang tiles.

Proposition 1. *The tile set T does not admit a periodic tiling.*

Proof. Assume that $f: \mathbb{Z}^2 \to T$ is a doubly periodic tiling with horizontal period a and vertical period b. For $i \in \mathbb{Z}$, let n_i denote the sum of colors on the upper edges of tiles $f(1, i), f(2, i), \ldots, f(a, i)$. Because the tiling is horizontally periodic with period a, the 'carries' on the left edge of $f(1, i)$ and the right edge of $f(a, i)$ are equal. Therefore $n_{i+1} = q_i n_i$, where $q_i = 2$ if tiles of T_2 are used on row i and $q_i = \frac{2}{3}$ if tiles of $T_{2/3}$ are used.

Because the vertical period of tiling f is b,

$$n_1 = n_{b+1} = q_1 q_2 \ldots q_b \cdot n_1,$$

and because two tiles with 0's on their upper edges cannot be next to each other, $n_1 \neq 0$. So $q_1 q_2 \ldots q_b = 1$. This contradicts the fact that no non-empty product of 2's and $\frac{2}{3}$'s can be 1. $\square$

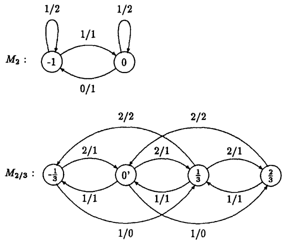


Fig. 2. Mealy machine corresponding to the aperiodic tile set.

Discoveries of aperiodic Wang tile sets (2015)

- Jeandel, Rao : every set of $\leq$ 10 tiles is **finite or periodic**
- Jeandel, Rao : an aperiodic set of **11** Wang tiles

Their algorithm is pictured below :

Image credit : Le Bagger 288, `http://i.imgur.com/YH9xX.jpg`

The transducer approach (Jeandel, Rao, 2015)

We identify a tile set $\mathcal{T}$ with its transducer (or its dual transducer).

Lemma

A Wang tile set $\mathcal{T}$ is **not aperiodic** if either

- (**finite**) there is k s.t. the str. conn. comp. of $\mathcal{T}^k$ is empty :
  i.e., there is no biinfinite words w, w' s.t. $w\mathcal{T}^k w'$.
- (**periodic**) or there exists k s.t. $\mathcal{T}^k$ is periodic :
  i.e., there is a biinfinite word w s.t. $w\mathcal{T}^k w$.

Jeandel, Rao (p. 8) :

The general algorithm to test for aperiodicity is therefore clear: for each k, generate $\mathcal{T}^k$, and test if one of the two situations happen. If it does, the set is not aperiodic. Otherwise, we go to the next k. The algorithm stops when the computer program runs out of memory. In that case, the algorithm was not able to decide if the Wang set was aperiodic (it is after all an undecidable problem), and we have to examine carefully this Wang set.

This approach works quite well in practice: when launched on a computer

Factor complexity

Let $w \in \mathcal{A}^{\mathbb{Z}}$. The **factor complexity** is a function $p_w(n) : \mathbb{N} \to \mathbb{N}$ counting the number of factors of length n in the sequence w.

$$w = ...000100\boxed{\mathbf{0100}}0100100010001000100100010001001...$$

$$Fact_w(4) = \{0001, 0010, \mathbf{0100}, 1000, 1001\} \implies p_w(4) = 5$$

Lemma

An infinite word $w \in \mathcal{A}^{\mathbb{Z}}$ that has $p_w(n) \leq n$ factors of length n is **periodic**.

Definition

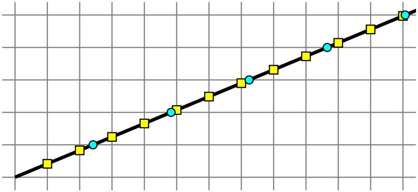
A **sturmian** word is an infinite word having exactly $p_w(n) = n+1$ factors of length n.

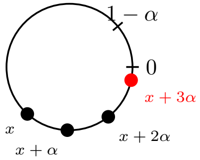
SYMBOLIC DYNAMICS II. STURMIAN TRAJECTORIES.*

By Marston Morse and Gustav A. Hedlund. (1940)

- **Sturmian sequences** encode the intersections of a irrational line with the grid.
- The **characteristic** (starting at origin) **Sturmian sequence** of slope $1/\sqrt{2}$ is :

$$0010010001001000100100100010010001001001\cdots$$



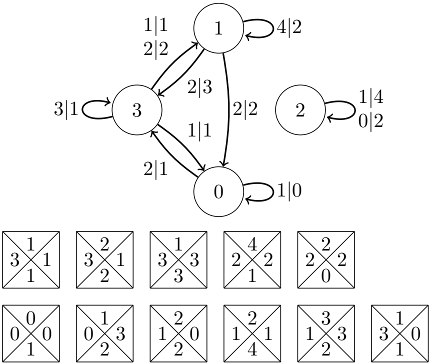


- They are obtained as coding of **irrational rotations on $\mathbb{R}/\mathbb{Z}$**.
- They can be **desubtituted** by $0\mapsto 0, 1\mapsto 01$ or $0\mapsto 1, 1\mapsto 01$.

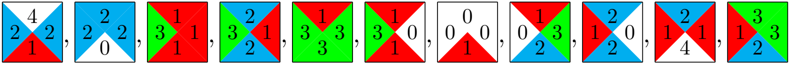
Jeandel, Rao, arxiv:1506.06492

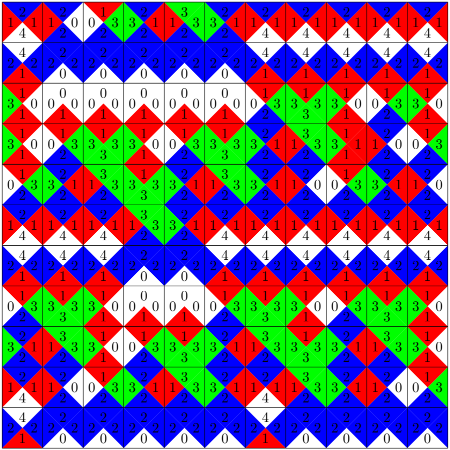
4 An aperiodic Wang set of 11 tiles - Proof Sketch

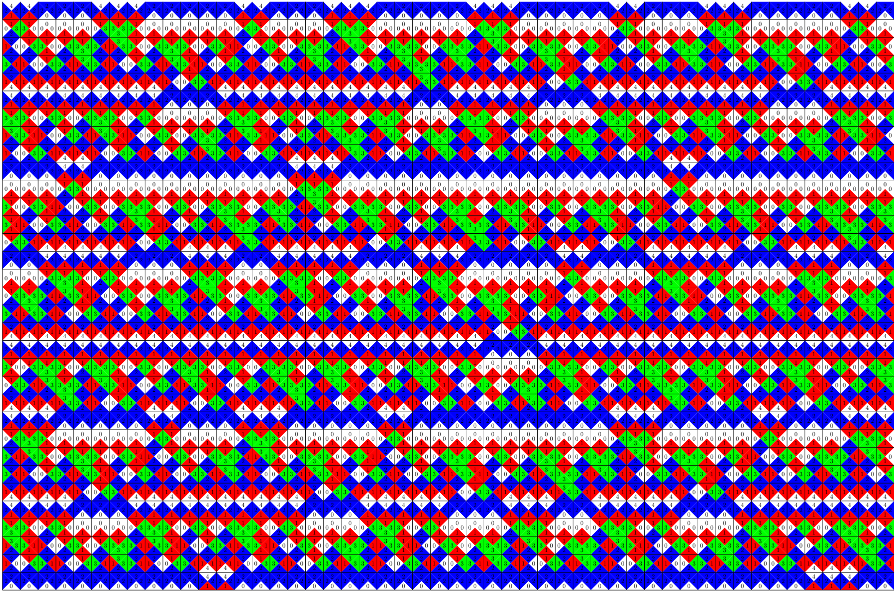
Using the same method presented in the last section, we were able to enumerate and test sets of 11 tiles, and found a few potential candidates. Of these few candidates, two of them were extremely promising and we will indeed prove that they are aperiodic sets.



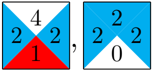
Jeandel-Rao 11 tiles set

$\mathcal{T} = \{$  $\}$.



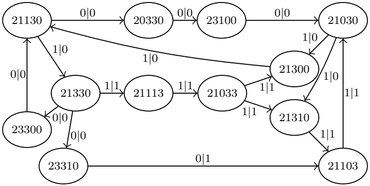


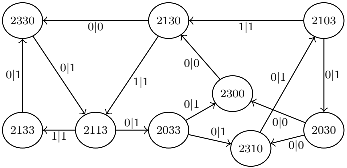
Some observations made by Jeandel, Rao

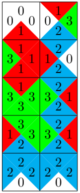
- horizontal lines are over $\mathcal{T}_1 = \Big\{$  $\Big\}$ or

$\mathcal{T}_0 = \Big\{$  $\Big\}$

- The str. conn. comp. of the product $\mathcal{T}_1\mathcal{T}_1$, $\mathcal{T}_1\mathcal{T}_0\mathcal{T}_1$, $\mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_1$ and $\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$ **are empty**.
- Every tiling by $\mathcal{T}$ can be **decomposed** into a tiling by transducers $\mathcal{T}_a = \mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$ and $\mathcal{T}_b = \mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$.
- Every tiling can be desubstituted uniquely by the **31 patterns** of rectangular shape 1×4 or 1×5 associated to the edges of $\mathcal{T}_a$ and $\mathcal{T}_b$:







Left image credit : `arxiv:1506.06492` , Figure 7 (b) at page 17.

An aperiodic set of 11 Wang tiles

Proposition (Jeandel, Rao, 2015)

The Wang set $\mathcal{T}_a \cup \mathcal{T}_b$ is **aperiodic**. Furthermore, the set of words $u \in \{a, b\}^*$ s.t. the sequence of transducers $\mathcal{T}_{u_1} \dots \mathcal{T}_{u_n}$ appear in a tiling of the plane is **exactly the set of factors of the Fibonacci word** (i.e., the fixed point of the morphism $a \mapsto ab, b \mapsto a$.

Theorem (Jeandel, Rao, 2015)

The 11 Wang tile set $\mathcal{T}$ is **aperiodic**.

Some remarks/questions :

- The proof is based **on transducers**.
- It seems there is **much more to understand**.
- Can we find a **short 10 lines proof** of existence and aperiodicity ?

Our contribution

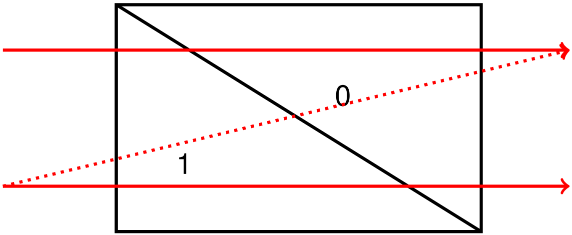
- We construct Jeandel-Rao tilings as codings of $\mathbb{Z}^2$-action on a torus,
- We conjecture this is a complete characterization.
- We construct a set of 19 Wang tiles $\mathcal{U}$ such that the set of all valid tilings $\mathbb{Z}^2 \to \mathcal{U}$ is self-similar, aperiodic and minimal.

In last slide of Michaël Rao's talk, Spring, 2017

Michaël wrote an observation that **is not** in their preprint :

> *"Open question 2 : "proof from the book" ? If we look at densities of 1 on each line on an infinite tiling, one tranducer* ***add*** $\varphi - 1$ *and the other* ***add*** $\varphi - 2$."

His remark made me think about **codings of rotations** :



Michaël accepted to give me a 2583×986 **patch** of $a's$ and $b's$ for which I am very thankful.

Wang tiles from codings of $\mathbb{Z}^2$-actions

- Let D be a **set**,
- $u, v : D \to D$ two **invertible transformations** s.t. $u \circ v = v \circ u$,
- I and J : two finite not necessarily disjoint **sets of colors**,
- $D = \cup_{i \in I} X_i$ and $D = \cup_{j \in J} Y_j$ be two **partitions** of D.

This gives the **left and bottom colors** :

$$\begin{array}{rccl} \ell : & D & \to & I \\ & \mathbf{x} & \mapsto & i \text{ if } \mathbf{x} \in X_i, \end{array} \qquad \begin{array}{rccl} b : & D & \to & J \\ & \mathbf{x} & \mapsto & j \text{ if } \mathbf{x} \in Y_j. \end{array}$$

and the **right and top colors** $r : D \to I$, $t : D \to J$ as :

$$r = \ell \circ u \quad \text{and} \quad t = b \circ v,$$

that is, the right color of an element $\mathbf{x} \in D$ is the left color of $u(\mathbf{x})$.
The **Wang tile coding** :

$$\begin{array}{rccl} c : & D & \to & I \times J \times I \times J \\ & \mathbf{x} & \mapsto & (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x})). \end{array}$$

Let $\mathcal{T} = c(D)$ be the associated **Wang tile set**.

Wang tilings from codings of $\mathbb{Z}^2$-actions

Let $c : D \to \mathcal{T}$ s.t. $c(\mathbf{x}) = (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x}))$.

Lemma

Let

$$\begin{array}{rccl} f: & D & \to & \mathcal{T}^{\mathbb{Z}^2} \\ & \mathbf{x} & \mapsto & f_{\mathbf{x}} : (m,n) \mapsto c(u^m v^n \mathbf{x}). \end{array}$$

For every $\mathbf{x} \in D$, $f_{\mathbf{x}}$ is a **Wang tiling of the plane**.
By definition f is a **conjugacy** between $\mathbb{Z}^2$-action of u and v on D and $\mathbb{Z}^2$-translations of the tilings.

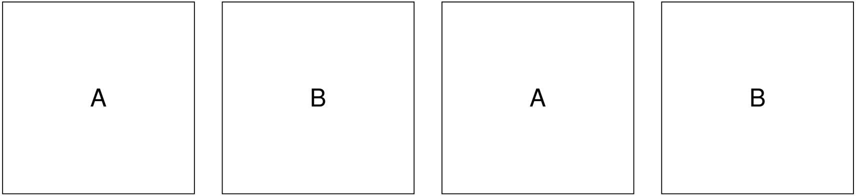
Therefore, unique ergodicity of the $\mathbb{Z}^2$-action on D will imply uniform patch frequencies in the tilings generated by f.

Codings of $\mathbb{Z}^2$-actions : Example 0

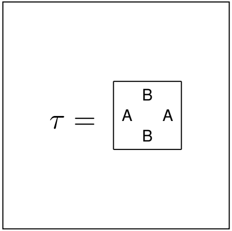
Let $\alpha, \beta \in \mathbb{R}$. On the **torus** $\mathbb{R}^2/\mathbb{Z}^2$, we consider the **translations**

$$u(\mathbf{x}) = \mathbf{x} + (\alpha, 0) \quad \text{and} \quad v(\mathbf{x}) = \mathbf{x} + (0, \beta)$$

right $r(\mathbf{x})$ top $t(\mathbf{x})$ left $\ell(\mathbf{x})$ bottom $b(\mathbf{x})$



$c(\mathbf{x}) = (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x}))$



- For every $\mathbf{x} \in \mathbb{R}^2/\mathbb{Z}^2$,

$$f_{\mathbf{x}} : \mathbb{Z}^2 \to \{\tau\}$$

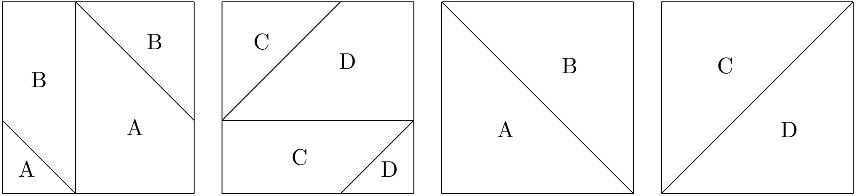
  is a periodic **Wang tiling of the plane**
- $c(\mathbb{R}^2/\mathbb{Z}^2) = \{\tau\}$ admits periodic tilings

Codings of $\mathbb{Z}^2$-actions : Example 1

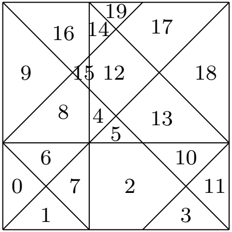
Let $\varphi = \frac{1+\sqrt{5}}{2}$. On the **torus** $\mathbb{R}^2/\mathbb{Z}^2$, we consider the **translations**

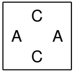
$$u(\mathbf{x}) = \mathbf{x} + (\varphi, 0) \quad \text{and} \quad v(\mathbf{x}) = \mathbf{x} + (0, \varphi)$$

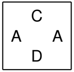
right $r(\mathbf{x})$ | top $t(\mathbf{x})$ | left $\ell(\mathbf{x})$ | bottom $b(\mathbf{x})$

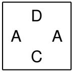


$c(\mathbf{x}) = (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x}))$



$\tau_0 =$ 

, $\tau_1 =$ 

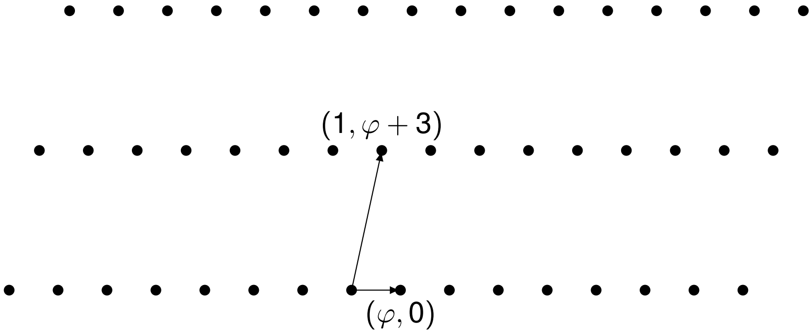
, $\tau_4 =$ 

, etc.

- For every $\mathbf{x} \in \mathbb{R}^2/\mathbb{Z}^2$, $f_\mathbf{x} : \mathbb{Z}^2 \to \mathcal{T}$ is a nonperiodic **Wang tiling of the plane**
- $c(\mathbb{R}^2/\mathbb{Z}^2)$ admits periodic tilings

Codings of $\mathbb{Z}^2$-actions : Example 3

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Consider the **lattice** $\Gamma = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$.



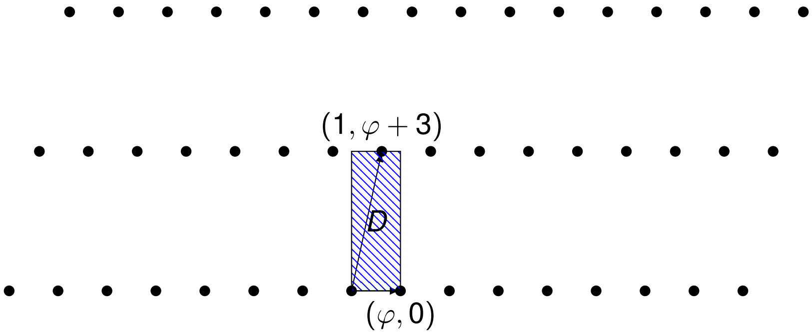
On the **torus** $\mathbb{R}^2/\Gamma$, we consider the **translations**

$$u : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x+1, y) \end{array} \quad \text{and} \quad v : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x, y+1). \end{array}$$

Codings of $\mathbb{Z}^2$-actions : Example 3

A **fundamental domain** of $\mathbb{R}^2/\Gamma$ is

$$D = [0, \varphi[\times [0, \varphi + 3[.$$

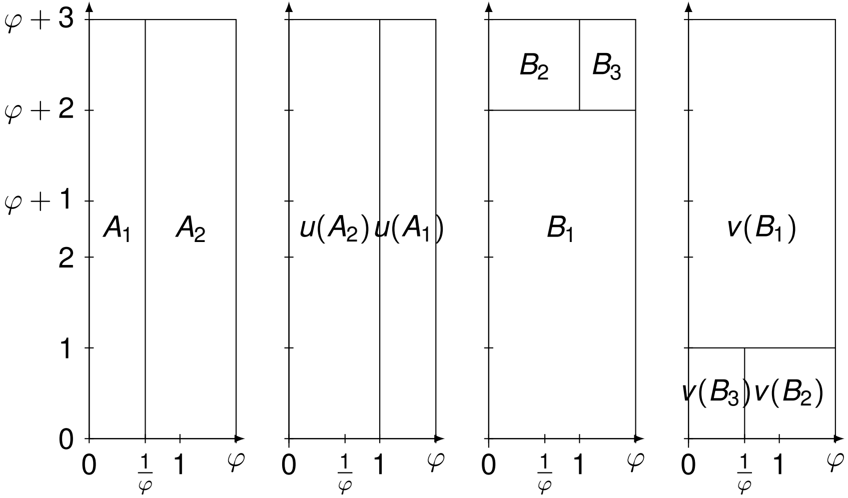


After renormalisation of transformations *u* and *v* on the torus $\mathbb{R}^2/\mathbb{Z}^2$, we observe that each translation vect. is not rationally independent :

$$\begin{pmatrix} \phi & 1 \\ 0 & \phi + 3 \end{pmatrix}^{-1} = \begin{pmatrix} \phi - 1 & -\frac{4}{11}\phi + \frac{5}{11} \\ 0 & -\frac{1}{11}\phi + \frac{4}{11} \end{pmatrix}.$$

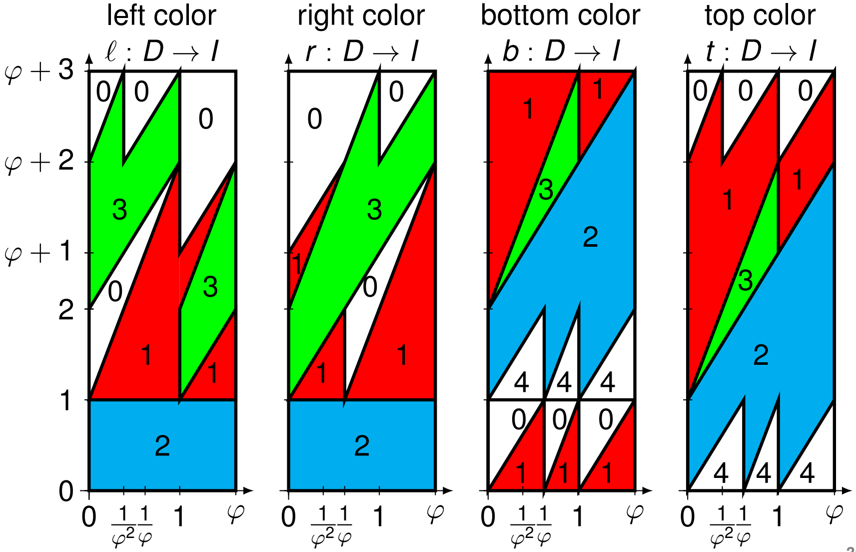
Codings of $\mathbb{Z}^2$-actions : Example 3

Transformations u and v are one-to-one **piecewise translations** of pieces on the fundamental domain D.



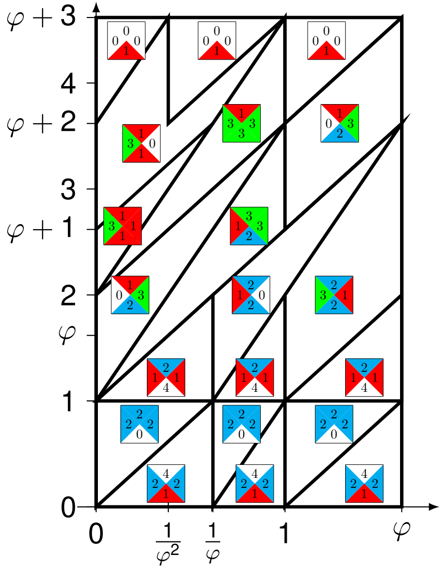
Codings of $\mathbb{Z}^2$-actions : Example 3

The **left, right bottom and top color codings** satisfying $r = \ell \circ u$ and $t = b \circ v$.



Codings of $\mathbb{Z}^2$-actions : Example 3

We deduce the **tile coding** $c : D \to \mathcal{T}$.



Theorem

We have

$$c(\mathbb{R}^2/\Gamma) = \mathcal{T}$$

where $\mathcal{T}$ is the **Jeandel-Rao tile set**.

Theorem

For every $\mathbf{x} \in \mathbb{R}^2/\Gamma$,

$$f_{\mathbf{x}} : \mathbb{Z}^2 \to \mathcal{T}$$

is a **Jeandel-Rao Wang tiling of the plane**.

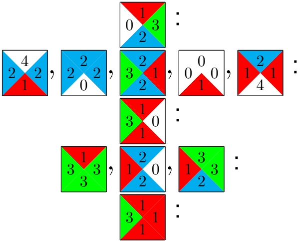
Frequency of patterns

Corollary

Since **Lebesgue** measure is the **only invariant** mesure on $\mathbb{R}^2/\Gamma$ which is invariant under both translations u and v, we have **unique ergodicity** of the tiling space

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}}$$

from which we deduce existence of pattern frequencies.



$$5/(12\varphi + 14) \approx 0.1496$$

$$1/(2\varphi + 6) \approx 0.1083$$

$$1/(5\varphi + 4) \approx 0.0827$$

$$1/(8\varphi + 2) \approx 0.0669$$

$$1/(18\varphi + 10) \approx 0.0256$$

Work in progress : towards a complete description

Let $\Omega_{\mathcal{T}}$ be the **Wang subshift** of all Wang tilings made of the Jeandel-Rao tile set $\mathcal{T}$.

Conjecture

Any Jeandel-Rao tiling in $\Omega_{\mathcal{T}}$ can be **desubstituted uniquely** into a tiling in the self-similar aperiodic tilings in $\Omega_{\mathcal{U}}$ where $\mathcal{U}$ is a set of 19 Wang tiles.

A sequence of 7 substitutions of the form $\square \mapsto \square, \square \mapsto \begin{smallmatrix}\square\\\square\end{smallmatrix}$ or $\square \mapsto \square, \square \mapsto \square\square$.

Conjecture

The symbolic dynamical system $(\Omega_{\mathcal{T}}, \sigma)$ where σ is the 2d shift is **measurably conjugate** to a $\mathbb{Z}^2$-rotation on the torus $\mathbb{R}^2/\mathbb{Z}^2$.

arXiv.org > math > arXiv:1802.03265

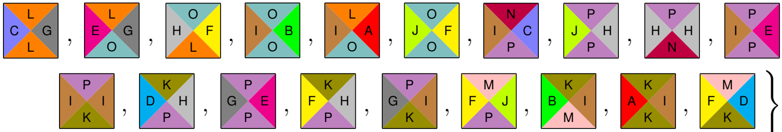
Mathematics > Dynamical Systems

A self-similar aperiodic set of 19 Wang tiles

Sébastien Labbé

(Submitted on 9 Feb 2018)

$\mathcal{U} = \{$



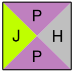
$\}$

Theorem

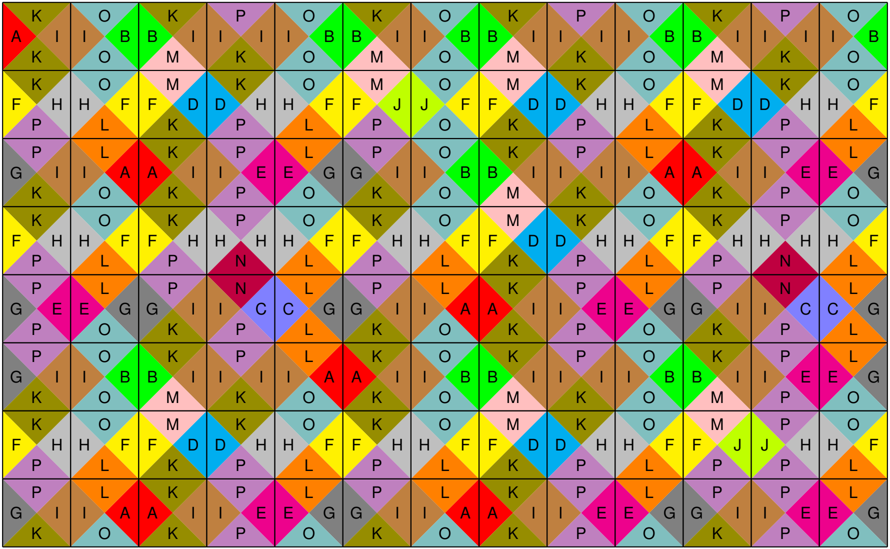
The Wang shift $\Omega_{\mathcal{U}}$ is **self-similar**, **aperiodic** and **minimal**.

$$\Omega_{\mathcal{U}} \xleftarrow{\alpha : \square \mapsto \square, \square \mapsto \begin{smallmatrix}\square\\\square\end{smallmatrix}} \Omega_{\mathcal{V}} \xleftarrow{\beta : \square \mapsto \square, \square \mapsto \square\square} \Omega_{\mathcal{W}} \xleftarrow{\gamma : \square \mapsto \square} \Omega_{\mathcal{U}}$$

Some patch in $\Omega_{\mathcal{U}}$

Let $\omega = \alpha\beta\gamma$ and $u_7 =$ 

, then $\omega^5(u_7) =$



The substitution ω

In terms of the $\{u_i\}_{0 \le i \le 18}$, the substitution $\sigma \circ \mu$ is

$$
\begin{array}{llll}
u_0 \mapsto \begin{pmatrix} u_{18} \\ u_{16} \end{pmatrix}, & u_1 \mapsto \begin{pmatrix} u_{18} \\ u_{17} \end{pmatrix}, & u_2 \mapsto \begin{pmatrix} u_{16} \end{pmatrix}, & u_3 \mapsto \begin{pmatrix} u_{13} \\ u_{14} \end{pmatrix}, \\
u_4 \mapsto \begin{pmatrix} u_{15} \\ u_{14} \end{pmatrix}, & u_5 \mapsto \begin{pmatrix} u_{17} \end{pmatrix}, & u_6 \mapsto \begin{pmatrix} u_{15} & u_5 \\ u_{14} & u_3 \end{pmatrix}, & u_7 \mapsto \begin{pmatrix} u_{17} & u_3 \end{pmatrix}, \\
u_8 \mapsto \begin{pmatrix} u_{16} & u_3 \end{pmatrix}, & u_9 \mapsto \begin{pmatrix} u_{13} & u_2 \\ u_{14} & u_4 \end{pmatrix}, & u_{10} \mapsto \begin{pmatrix} u_{13} & u_2 \\ u_{12} & u_1 \end{pmatrix}, & u_{11} \mapsto \begin{pmatrix} u_{14} & u_3 \end{pmatrix}, \\
u_{12} \mapsto \begin{pmatrix} u_{11} & u_2 \\ u_{10} & u_4 \end{pmatrix}, & u_{13} \mapsto \begin{pmatrix} u_{10} & u_3 \end{pmatrix}, & u_{14} \mapsto \begin{pmatrix} u_{11} & u_2 \\ u_9 & u_1 \end{pmatrix}, & u_{15} \mapsto \begin{pmatrix} u_{10} & u_4 \end{pmatrix}, \\
u_{16} \mapsto \begin{pmatrix} u_8 & u_2 \\ u_6 & u_0 \end{pmatrix}, & u_{17} \mapsto \begin{pmatrix} u_7 & u_2 \\ u_9 & u_1 \end{pmatrix}, & u_{18} \mapsto \begin{pmatrix} u_9 & u_1 \end{pmatrix}. &
\end{array}
$$

Lemma

ω is primitive. Its characteristic polynomial is

$$\chi_M(x) = x^3 \cdot (x-1)^4 \cdot (x+1)^4 \cdot (x^2+x-1)^3 \cdot (x^2-3x+1).$$

The PF eigenvalue is $\varphi^2 = \varphi + 1 = (3+\sqrt{5})/2$.

The Open Questions

- Find the **10 line proof** for the aperiodicity of Jeandel-Rao tilings.
- Can we generalize Jeandel-Rao tilings to **other Pisot numbers** ?
- I think the fundamental domain can be identified with the **window** of a cut-and-project set with dimension $2 + 2$.
- What are the structure of the **other aperiodic tile sets** of cardinality 11 found by Jeandel-Rao ?
- Does there **exists an aperiodic self-similar** Wang tile set of cardinality less than 16 ?
- Generalize the **characterization of primitive sequences** by Durand (1998) to Wang shifts.
- Describe the space of all aperiodic Wang shifts of low complexity (are **continued fraction** algorithms involved like for Sturmian sequences ?)

The Open Questions

- Understand all of this in terms of **shape-symmetric** multidim. sequences (Maes, 1999) and S-automatic sequences using

Theorem (Charlier, Kärki, Rigo, 2010)

Let $d \geq 1$. The d-dimensional infinite word x is **S-automatic** for some abstract numeration system $S = (L, \Sigma, <)$ where $\epsilon \in L$ if and only if x is the **image by a coding of a shape-symmetric** d-dimensional infinite word.

- For yesterdays speakers : can we find a model (based on voronoi cells, or typical quadrangulations) such that atoms have "liquid" structure when $\alpha < \frac{3}{2}$ organize as a quasicrystals when $\alpha > \frac{3}{2}$?

Sage code

I wrote **three Python modules** while working on this project :

```
sage: from slabbe import WangTileSet, WangTileSolver
sage: from slabbe import Substitution2d
sage: from slabbe import PolyhedronPartition
```

The code is **open-source** :

```
https://github.com/seblabbe/slabbe
```

It is part of the optional Sage package `slabbe-0.4.1` which can **install** with :

```
sage -pip install slabbe
```

17th Mons Theoretical Computer Science Days

Journées montoises : `http://jm2018.sciencesconf.org/`

September 10-14, 2018 at LaBRI

SCOPE :

- **word combinatorics** and formal languages from their **different perspectives** (combinatorial, algorithmic, dynamical, logic, ...).
- **welcomes other branches** of computer science and mathematics linked to it (number theory, computability, model checking, semigroups, game theory, discrete geometry, decentralized algorithms, bioinformatics, ...).

SUBMISSION :

- Abstracts between **1 and 4 pages**

IMPORTANT DATES :

- submission before **May 28th 2018**