Indistinguishable asymptotic pairs and multidimensional Sturmian configurations

arXiv:2204.06413

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18e Journées Montoises d'Informatique Théorique
Prague, September 5-9, 2022

Outline

1. **A question on asymptotic pairs of configurations**
2. **When $d = 1$: a complete answer**
3. **When $d \geq 1$: a partial answer**
4. **Proofs (some ideas involved)**
5. **Open questions**

Outline

Configurations

A map $x : \mathbb{Z}^d \to \Sigma$ is called a **configuration**.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$\sigma^{(4,1)}x$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

(both restricted to $[-5,5] \times [-4,3]$)

The **shift action** $\mathbb{Z}^d \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}^d}$ is given by the map $\sigma : \mathbb{Z}^d \times \Sigma^{\mathbb{Z}^d} \to \Sigma^{\mathbb{Z}^d}$ where

$$\sigma^u(x)_v := \sigma(u,x)_v = x_{u+v} \quad \text{for every } u, v \in \mathbb{Z}^d, x \in \Sigma^{\mathbb{Z}^d}.$$

Asymptotic pair

Two configurations $x, y \in \Sigma^{\mathbb{Z}^d}$ are **asymptotic** if they differ in finitely many sites of $\mathbb{Z}^d$.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$\sigma^{(4,1)}x$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

(both restricted to $[-5, 5] \times [-4, 3]$)

The set $F = \{\boldsymbol{n} \in \mathbb{Z}^d : x_{\boldsymbol{n}} \neq y_{\boldsymbol{n}}\}$ is called the **difference set** of (x, y).

Pattern, Cylinder, Language, Occurrences

- For finite subset $S \subset \mathbb{Z}^d$, an function $p\colon S \to \Sigma$ is called a **pattern** and the set S is its **support**. We denote it $p \in \Sigma^S$.
- Given a pattern $p \in \Sigma^S$, the **cylinder** centered at p is

$$[p] = \{x \in \Sigma^{\mathbb{Z}^d} : x|_S = p\}.$$

- For finite subset $S \subset \mathbb{Z}^d$, the **language with support S** of a configuration x is the set of patterns

$$\mathcal{L}_S(x) = \{p \in \Sigma^S : \text{ there is } \boldsymbol{n} \in \mathbb{Z}^d \text{ such that } \sigma^{\boldsymbol{n}}(x) \in [p]\}.$$

  The **language of x** is the union $\mathcal{L}(x)$ of the sets $\mathcal{L}_S(x)$ for every finite $S \subset \mathbb{Z}^d$.
- The **occurrences** of a pattern $p \in \Sigma^S$ in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ is

$$\mathrm{occ}_p(x) := \{\boldsymbol{n} \in \mathbb{Z}^d : \sigma^{\boldsymbol{n}}(x) \in [p]\}.$$

Language

$$x : \mathbb{Z}^2 \to \{0, 1, 2\}$$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The **language** of patterns of support $S = \{\mathbf{0}, \boldsymbol{e}_1, 2\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in x is

$$\mathcal{L}_S(x) = \left\{ \begin{matrix} \begin{smallmatrix} 0 & & \\ 2 & 1 & 0 \end{smallmatrix}, & \begin{smallmatrix} 0 & & \\ 2 & 1 & 1 \end{smallmatrix}, & \begin{smallmatrix} 1 & & \\ 0 & 2 & 1 \end{smallmatrix}, & \begin{smallmatrix} 1 & & \\ 2 & 2 & 1 \end{smallmatrix}, \\ \begin{smallmatrix} 1 & & \\ 0 & 2 & 2 \end{smallmatrix}, & \begin{smallmatrix} 2 & & \\ 0 & 2 & 2 \end{smallmatrix}, & \begin{smallmatrix} 2 & & \\ 1 & 0 & 2 \end{smallmatrix}, & \begin{smallmatrix} 2 & & \\ 1 & 1 & 0 \end{smallmatrix} \end{matrix} \right\}$$

Also $\mathcal{L}_S(x, y) := \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$ for every finite support $S \subset \mathbb{Z}^d$.

Occurrences within asymptotic pairs

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The occurrences of the pattern $p =$

1		
0	2	1

in x and y are

$$\mathrm{occ}_p(x) = \{(-5, 2), (1, 1), (3, -3), \ldots\},$$
$$\mathrm{occ}_p(y) = \{(-5, 2), (1, 1), (3, -3), \ldots\},$$

Occurrences within asymptotic pairs

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The occurrences of the pattern $p =$

1		
0	2	1

in x and y are

$$\begin{aligned} \mathrm{occ}_p(x) &= \{(-5,2),(1,1),(3,-3),\ldots\}, \\ \mathrm{occ}_p(y) &= \{(-5,2),(1,1),(3,-3),\ldots\}, \\ \mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y) &= \{(0,0)\}, \\ \mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x) &= \{(-2,-1)\}. \end{aligned}$$

Indistinguishable asymptotic pair

Let $p \in \Sigma^S$ is a pattern of finite support $S \subset \mathbb{Z}^d$.

If $x, y \in \Sigma^{\mathbb{Z}^d}$ are asymptotic configurations with difference set F, then

$$\text{occ}_p(x) \setminus \text{occ}_p(y) = \text{occ}_p(x) \cap (F - S)$$

and in particular it **is finite**.

Definition

We say that (x, y) is an **indistinguishable asymptotic pair** if (x, y) is asymptotic and the following equality holds

$$\#\left(\text{occ}_p(x) \setminus \text{occ}_p(y)\right) = \#\left(\text{occ}_p(y) \setminus \text{occ}_p(x)\right)$$

for every pattern p of finite support.

Not all asymptotic pair is indistinguishable

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

The occurrences of the pattern $p =$

1	
	2

in x and y are

$$\begin{aligned} \mathrm{occ}_p(x) &= \{(-1,-1)\}, \\ \mathrm{occ}_p(y) &= \varnothing, \\ \mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y) &= \{(-1,-1)\}, \\ \mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x) &= \varnothing. \end{aligned}$$

Initial question

In Fall 2019, during a visit to Prague, Sebastian Barbieri asked :

Question

Is there any non trivial pair $x, y \in \Sigma^{\mathbb{Z}^d}$ of indistinguishable asymptotic configurations ?

A trivial pair refers to cases like (x, x) and $(x, \sigma^{\boldsymbol{n}}(x))$ where $\boldsymbol{n} \in \mathbb{Z}^d$.

S. Barbieri, R. Gómez, B. Marcus, T. Meyerovitch, and S. Taati. Gibbsian representations of continuous specifications : the theorems of Kozlov and Sullivan revisited. Communications in Mathematical Physics*, 382(2) :1111–1164, 2021.*

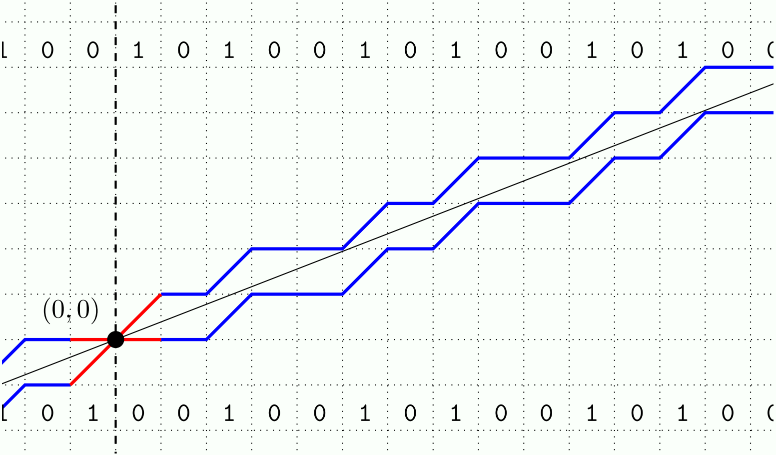
Outline

Sturmian words (Morse, Hedlund, 1940)

Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and $c_\alpha, c'_\alpha : \mathbb{Z} \to \{0,1\}$ be the configurations

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor \quad \text{(\textbf{lower} characteristic Sturmian word)}$$

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil \quad \text{(\textbf{upper} characteristic Sturmian word)}$$



... are indistinguishable

$$c_\alpha = \cdots 101001010010 \boxed{1.0} 010010100101 \cdots$$
$$c'_\alpha = \cdots 101001010010 \boxed{0.1} 010010100101 \cdots$$

... are indistinguishable and vice versa

$$c_\alpha = \cdots 101001010010\boxed{1.0}010010100101\cdots$$
$$c'_\alpha = \cdots 101001010010\boxed{0.1}010010100101\cdots$$

Theorem (Barbieri, L, Starosta, 2021)

*Let $x, y \in \{0,1\}^{\mathbb{Z}}$ and assume that x is **recurrent**.*
*The pair (x, y) is an **indistinguishable asymptotic pair** with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$*

if and only if

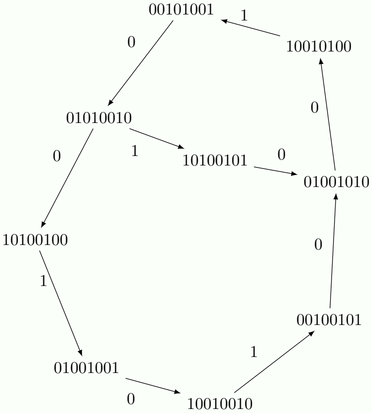
*there exists $\alpha \in [0,1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper **characteristic Sturmian words** of slope α.*

*Barbieri, L., Starosta, A characterization of Sturmian sequences by indistinguishable asymptotic pairs, European Journal of Combinatorics **95** (2021) 103318,* doi:10.1016/j.ejc.2021.103318

Application : Markov injectivity conjecture

$$x = \cdots 101001010010\boxed{1.0}010010100101\cdots$$

$$y = \cdots 101001010010\boxed{0.1}010010100101\cdots$$



1010010
00100101
00101001
01001001
01001010
01010010
10010010
10010100
10100100
10100101
001001010

*L., Lapointe, The q-analog of the Markoff injectivity conjecture over the language of a balanced sequence, Combinatorial Theory **2** (2022) #9, 25 pages.*
doi:10.5070/C62156881

Outline

Flip condition

Definition

An asymptotic pair $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ satisfies the **flip condition** if

1. the difference set of x and y is $F = \{\mathbf{0}, -\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_d\}$,
2. the restriction $x|_F$ is a **bijection** $F \to \{0, 1, \ldots, d\}$,
3. the map defined by $x_{\boldsymbol{n}} \mapsto y_{\boldsymbol{n}}$ for every $\boldsymbol{n} \in F$ is a **cyclic permutation** on the alphabet $\{0, 1, \ldots, d\}$.

Without lost of generality, we assume that $x_{\mathbf{0}} = 0$ and $y_{\boldsymbol{n}} = x_{\boldsymbol{n}} - 1 \mod (d+1)$ for every $\boldsymbol{n} \in F$.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Flip condition

Definition

An asymptotic pair $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ satisfies the **flip condition** if

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   is a **cyclic permutation** on the alphabet $\{0, 1, \ldots, d\}$.

Without lost of generality, we assume that $x_{\mathbf{0}} = 0$ and
$y_{\boldsymbol{n}} = x_{\boldsymbol{n}} - 1 \mod (d+1)$ for every $\boldsymbol{n} \in F$.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

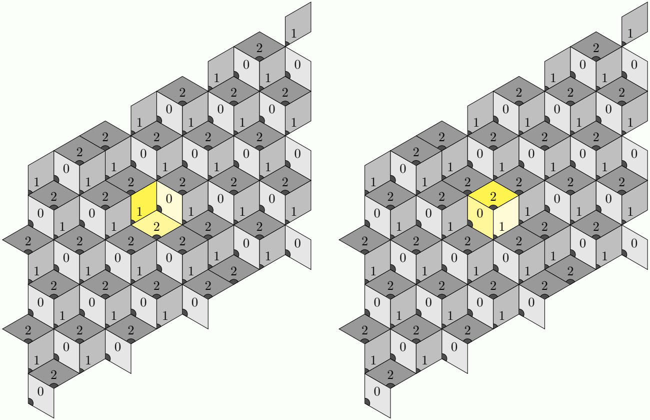
1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

Flip condition

The flip condition may be interpreted as the **geometrical flip** of the faces of a hypercube at the origin of a **discrete hyperplane** :



T. Jolivet. Combinatorics of Pisot Substitutions. *PhD Thesis, 2013.*

Damien Jamet, Coding Stepped Planes and Surfaces by Two-Dimensional Sequences over a Three-Letter Alphabet 05047, 2005, pp.21

Theorem A

Theorem (Barbieri, L., 2022)

*Let $d \geq 1$ and $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be an asymptotic pair satisfying the **flip condition** with difference set $F = \{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_d\}$. The following are equivalent :*

- (i) *For every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, we have*

$$\#\left(\mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y)\right) = 1 = \#\left(\mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x)\right).$$

- (ii) *The asymptotic pair (x, y) is **indistinguishable**.*
- (iii) *For every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$, the **pattern complexity** of x and y is*

$$\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S).$$

Theorem B

$\boldsymbol{\alpha} \in \mathbb{R}^d$ is a **totally irrational** vector if $\boldsymbol{n} \in \mathbb{Z}^d$, $\boldsymbol{n} \cdot \boldsymbol{\alpha} \in \mathbb{Z} \implies \boldsymbol{n} = \boldsymbol{0}$

Theorem (Barbieri, L., 2022)

*Let $d \geq 1$ and $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ s.t. x is **uniformly recurrent**. The pair (x, y) is an **indistinguishable asymptotic pair** satisfying the flip condition*

if and only if

*there exists a **totally irrational vector** $\boldsymbol{\alpha} \in [0, 1)^d$ such that $x = c_{\boldsymbol{\alpha}}$ and $y = c'_{\boldsymbol{\alpha}}$ are the lower and upper **characteristic d-dimensional Sturmian configurations** with slope $\boldsymbol{\alpha}$.*

$$\begin{aligned} c_{\boldsymbol{\alpha}} : \quad \mathbb{Z}^d &\to \{0, 1, \ldots, d\} \\ \boldsymbol{n} &\mapsto \sum_{i=1}^{d} \left(\lfloor \alpha_i + \boldsymbol{n} \cdot \boldsymbol{\alpha} \rfloor - \lfloor \boldsymbol{n} \cdot \boldsymbol{\alpha} \rfloor \right) \\ c'_{\boldsymbol{\alpha}} : \quad \mathbb{Z}^d &\to \{0, 1, \ldots, d\} \\ \boldsymbol{n} &\mapsto \sum_{i=1}^{d} \left(\lceil \alpha_i + \boldsymbol{n} \cdot \boldsymbol{\alpha} \rceil - \lceil \boldsymbol{n} \cdot \boldsymbol{\alpha} \rceil \right). \end{aligned}$$

Example

With $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (\sqrt{2}/2, \sqrt{19} - 4)$:

$c_{\boldsymbol{\alpha}} : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$c'_{\boldsymbol{\alpha}} : \mathbb{Z}^2 \to \{0, 1, 2\}$

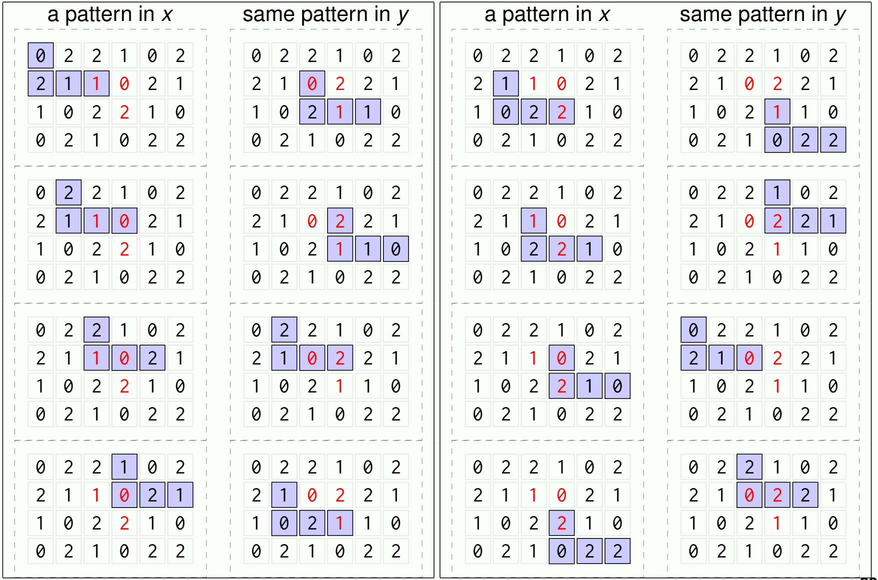
1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

is an indistinguishable asymptotic pair encoding the projection of the surface of a discrete plane of normal vector

$$(1 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2) \approx (0.293, 0.348, 0.359).$$

Example

The 8 patterns of support $\{\mathbf{0}, \boldsymbol{e}_1, 2\boldsymbol{e}_1, \boldsymbol{e}_2\}$ appearing in x and y :



Complexity $\#(F - S)$

Complexity $\#(F - S)$ matches what is known :

- When $d = 1$ and $S = \{0, 1, \ldots, n-1\}$:

$$\#(F - S) = \#\left(\{0, -1\} - \{0, 1, \ldots, n-1\}\right) = n + 1$$

  is the factor complexity of **Sturmian words**.

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Complexity $\#(F - S)$ matches what is known :

- When $d = 1$ and $S = \{0, 1, \ldots, n-1\}$:

$$\#(F - S) = \#\left(\{0, -1\} - \{0, 1, \ldots, n-1\}\right) = n + 1$$

  is the factor complexity of **Sturmian words**.

- When $d = 2$ and $S = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$:,

$$\begin{aligned}\#(F - S) &= \#\left(\{\mathbf{0}, -\boldsymbol{e}_1, -\boldsymbol{e}_2\} - \{(i, j) \colon 0 \leq i < n, 0 \leq j < m\}\right) \\ &= mn + m + n\end{aligned}$$

  is the rectangular pattern complexity of a **discrete plane** with totally irrational (irrational and rationally independent) slope.

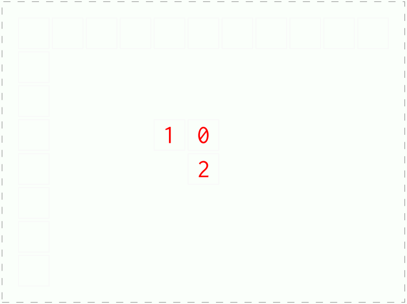
V. Berthé, L. Vuillon. Tilings and rotations on the torus : a two-dimensional generalization of Sturmian sequences. Discrete Mathematics, *223(1-3) :27–53, 2000.*

Outline

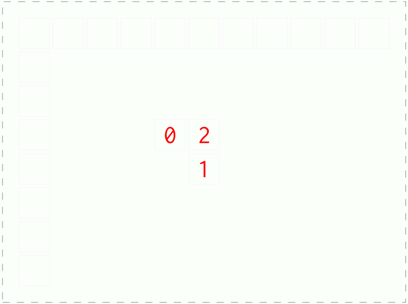
Proof of Theorem A. (ii) $\implies$ (iii)

$\#\mathcal{L}_S(x) \leq \#(F - S)$:

x

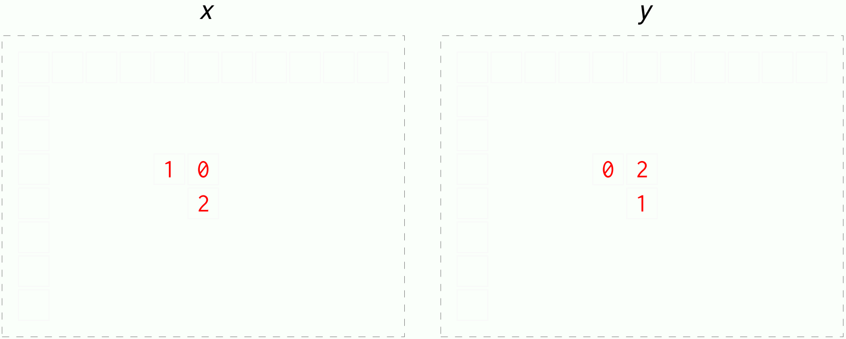


y



Proof of Theorem A. (ii) $\implies$ (iii)

$\#\mathcal{L}_S(x) \leq \#(F - S)$:



$\#\mathcal{L}_S(x) \geq \#(F - S)$:

we skip this proof, see the preprint.

Special factors in higher dimensions

The **extensions at position $\ell \in \mathbb{Z}^d \setminus S$** of the pattern $w \in \mathcal{L}_S(x, y)$ is

$$E^\ell(w) := \{u_\ell \colon u \in \mathcal{L}_{S\cup\{\ell\}}(x, y) \text{ and } u|_S = w\}.$$

A pattern $w \in \mathcal{L}_S(x, y)$ is **special at position $\ell \in \mathbb{Z}^d$** if $\#E^\ell(w) \geq 2$.
The **bilateral extensions at positions $\ell, r \in \mathbb{Z}^d \setminus S$** of the pattern w within the language $\mathcal{L}(x, y)$ is

$$E^{\ell,r}(w) = \{(u_\ell, u_r) \colon u \in \mathcal{L}_{S\cup\{\ell,r\}}(x, y) \text{ and } u|_S = w\}.$$

The **bilateral multiplicity** $m^{\ell,r}(w)$ of the pattern w at the positions $\ell, r \in \mathbb{Z}^d \setminus S$ within the language $\mathcal{L}(x, y)$ is given by the expression

$$m^{\ell,r}(w) = \#E^{\ell,r}(w) - \#E^\ell(w) - \#E^r(w) + 1.$$

A pattern $w \in \mathcal{L}_S(x, y)$ is **strong** (resp. **weak**, **neutral**) at the positions $\ell, r \in \mathbb{Z}^d \setminus S$ if $m^{\ell,r}(w) > 0$ (resp. $m^{\ell,r}(w) < 0$, $m^{\ell,r}(w) = 0$).

Special factors in higher dimensions

The **bilateral extensions at positions $\ell, r \in \mathbb{Z}^d \setminus S$** of the pattern w within the language $\mathcal{L}(x, y)$

$$E^{\ell,r}(w) = \{(u_\ell, u_r) \colon u \in \mathcal{L}_{S \cup \{\ell,r\}}(x, y) \text{ and } u|_S = w\}.$$

can be interpreted as an undirected bipartite graph called **extension graph**.

V. Berthé, F. Dolce, F. Durand, J. Leroy, and D. Perrin. Rigidity and substitutive dendric words. Int. J. of Foundations of Computer Science, *29(5) :705–720, 2018.*

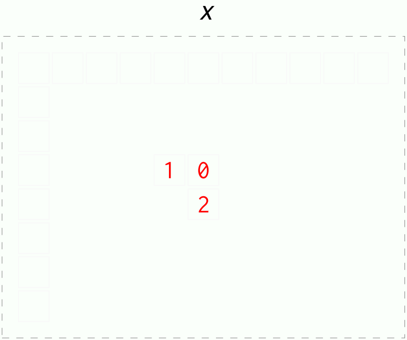
Lemma

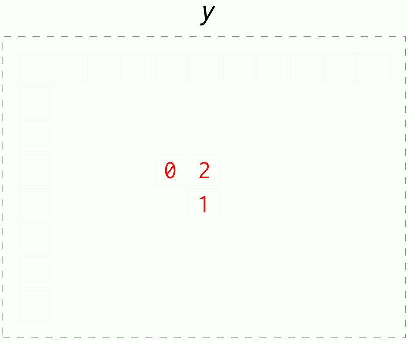
Let $w \in \mathcal{L}_S(x, y)$ be a pattern and c be the number of connected components of $E^{\ell,r}(w)$.

1. $m^{\ell,r}(w) \geq 1 - c$.
2. *The extension graph $E^{\ell,r}(w)$ is acyclic iff $m^{\ell,r}(w) = 1 - c$.*
3. *If $E^{\ell,r}(w)$ is connected, then $m^{\ell,r}(w) \geq 0$.*
4. *If $E^{\ell,r}(w)$ is connected and contains a cycle, then $m^{\ell,r}(w) > 0$.*

Proof of Theorem A. (iii) $\implies$ (i)

We assume that for every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$, the **pattern complexity** of x and y is $\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S)$. Suppose by contradiction, that a pattern occurs twice intersecting F in x, then the extension graph of some pattern contains a cycle :

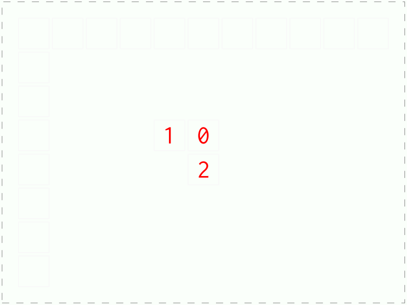




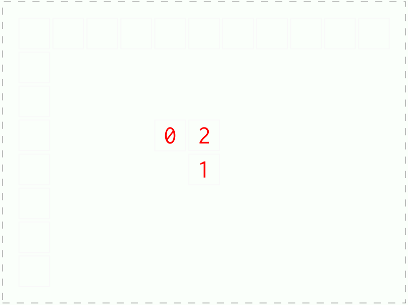
Proof of Theorem A. (iii) $\implies$ (i)

We assume that for every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$, the **pattern complexity** of x and y is $\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S)$. Suppose by contradiction, that a pattern occurs twice intersecting F in x, then the extension graph of some pattern contains a cycle :

x



y



This is a contradiction, because for every pattern $w \in \mathcal{L}_S(x, y)$ such that $S \cup \{\ell\}$ $S \cup \{r\}$ $S \cup \{\ell, r\}$ are connected, we show that $m^{\ell,r}(w) = 1 - c$ and thus the extension graph $E^{\ell,r}(w)$ is acyclic.

Proof of Theorem B

Proof is done by induction on the dimension d :

$$\begin{array}{rcl} \pi : \quad \{0, 1, \ldots, d\} & \to & \{0, \ldots, d-1\} \\ j & \mapsto & \begin{cases} 0 & \text{if } j = 0, \\ j - 1 & \text{if } j \neq 0. \end{cases} \end{array}$$

which extends to configurations $x \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ by letting

$$\pi(x) = (\pi(x_{\boldsymbol{n}}))_{\boldsymbol{n} \in \mathbb{Z}^d} \in \{0, \ldots, d-1\}^{\mathbb{Z}^d}.$$

Proposition

Let $d \geq 2$ be an integer. Let $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be an indistinguishable asymptotic pair satisfying the ordered flip condition. Then $\pi \circ x \circ \ell_{0,\boldsymbol{e}_1^\perp}$ and $\pi \circ y \circ \ell_{0,\boldsymbol{e}_1^\perp}$ are indistinguishable asymptotic configurations in $\{0, 1, \ldots, d-1\}^{\mathbb{Z}^{d-1}}$ which satisfy the ordered flip condition in dimension $d - 1$.

Proof of Theorem B

x

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

y

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

- If A_n (resp. B_n) is the language of horizontal words of length n at height 0 (resp. -1) in x and y, then $A_n \cap B_n \neq \varnothing$.
- Let $\pi_{2\to1}$ be the projection $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$.
- Every line in $\pi_{2\to1}(x)$ and $\pi_{2\to1}(y)$ lives in the same Sturmian subshift.

We consider their image on the circle under the factor map (intercept).

Proof of Theorem B

$\pi_{2\to1}(x)$

1	0	1	1	1	0	1	1	0	1	1
0	1	1	1	0	1	1	0	1	1	0
1	1	0	1	1	1	0	1	1	0	1
1	0	1	1	1	0	1	1	0	1	1
0	1	1	0	1	1	1	0	1	1	0
1	1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1	1
0	1	1	0	1	1	0	1	1	1	0

$\pi_{2\to1}(y)$

1	0	1	1	1	0	1	1	0	1	1
0	1	1	1	0	1	1	0	1	1	0
1	1	0	1	1	1	0	1	1	0	1
1	0	1	1	0	1	1	1	0	1	1
0	1	1	0	1	1	1	0	1	1	0
1	1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1	1
0	1	1	0	1	1	0	1	1	1	0

- If A_n (resp. B_n) is the language of horizontal words of length n at height 0 (resp. -1) in x and y, then $A_n \cap B_n \neq \varnothing$.
- Let $\pi_{2\to1}$ be the projection $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$.
- Every line in $\pi_{2\to1}(x)$ and $\pi_{2\to1}(y)$ lives in the same Sturmian subshift.

We consider their image on the circle under the factor map (intercept).

Proof of Theorem B

The sequence of intercepts changes by the same constant from one line to the next.

$\pi_{2\to1}(x)$

1	0	1	1	1	0	1	1	0	1	1
0	1	1	1	0	1	1	0	1	1	0
1	1	0	1	1	1	0	1	1	0	1
1	0	1	1	1	0	1	1	0	1	1
0	1	1	0	1	1	1	0	1	1	0
1	1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1	1
0	1	1	0	1	1	0	1	1	1	0

$\pi_{2\to1}(y)$

1	0	1	1	1	0	1	1	0	1	1
0	1	1	1	0	1	1	0	1	1	0
1	1	0	1	1	1	0	1	1	0	1
1	0	1	1	0	1	1	1	0	1	1
0	1	1	0	1	1	1	0	1	1	0
1	1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1	1
0	1	1	0	1	1	0	1	1	1	0

Outline

Open question 1

Theorem (Barbieri, L, Starosta, 2021)

Let $x, y \in \{0, 1\}^{\mathbb{Z}}$.
*The pair (x, y) is an **indistinguishable asymptotic pair** with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$*

if and only if

*there exists a monotone sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in [0, 1] \setminus \mathbb{Q}$ s.t. $x = \lim_{n\to\infty} c_{\alpha_n}$ and $y = \lim_{n\to\infty} c'_{\alpha_n}$ are the limits of **characteristic Sturmian words** of slope α_n.*

Moreover, indistinguishable asymptotic pairs over $\mathbb{Z}$ for any finite difference set F are described in terms of derived sequences.

Open question

Describe indistinguishable asymptotic pair $x, y \in \Sigma^{\mathbb{Z}^d}$ satisfying the flip condition where x is **not uniformly recurrent**.

Open question 2

A sequence $w \in \Sigma^{\mathbb{Z}}$ with $\#\mathcal{L}_n(w) \leq n$ is eventually periodic.

Nivat's conjecture

A configuration $x \in \Sigma^{\mathbb{Z}^2}$ for which there are $n, m \in \mathbb{N}$ with $\#\mathcal{L}_{(n,m)}(x) \leq nm$ is periodic.

Equivalently, a sequence $w \in \Sigma^{\mathbb{Z}}$ with totally irrational vector of symbol frequencies has complexity $\#\mathcal{L}_n(w) \geq n + 1$.

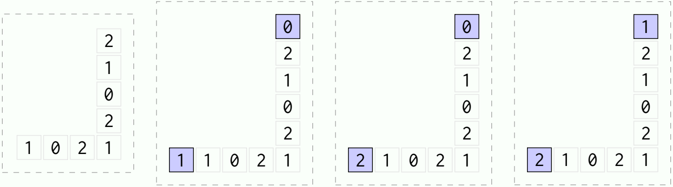
Dual Nivat Conjecture

Let $d \geq 1$ and $F = \{\mathbf{0}, -\boldsymbol{e}_1, \dots, -\boldsymbol{e}_d\}$. Let $x \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ be a configuration with trivial stabilizer, i.e., $\sigma^{\boldsymbol{n}}(x) = x$ only holds for $\boldsymbol{n} = \mathbf{0}$. If the **frequencies of symbols** in x exist and form a **totally irrational** vector, then $\#\mathcal{L}_S(x) \geq \#(F - S)$ for every nonempty connected finite support $S \subset \mathbb{Z}^d$.

J. Cassaigne. Double sequences with complexity $mn + 1$. volume 4, pages 153–170. 1999. Journées Montoises d'Informatique Théorique (Mons, 1998).

Open question 3

The pattern below is bispecial within the language of c_α and c'_α :



Bispecial factors within the language of a Sturmian sequence of slope $\alpha \in [0,1)$ are related to the convergents of the continued fraction expansion of α (de Luca, 1997).

Question

Let $d \geq 1$ and $\boldsymbol{\alpha} \in [0,1)^d$ be a totally irrational vector. What is the relation between the set

$$V_{\boldsymbol{\alpha}} = \left\{ b - a \colon \exists w \in \mathcal{L}(c_{\boldsymbol{\alpha}}) \text{ bispecial at positions } a, b \in \mathbb{Z}^d \right\}$$

and **simultaneous Diophantine approximations** of $\boldsymbol{\alpha}$?