Indistinguishable asymptotic pairs and multidimensional Sturmian configurations

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Outline

Outline

Configurations

A map $x : \mathbb{Z}^d \to \Sigma$ is called a **configuration**.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$\sigma^{(4,1)} x$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

(both restricted to $[-5, 5] \times [-4, 3]$)

The **shift action** $\mathbb{Z}^d \overset{\sigma}{\curvearrowright} \Sigma^{\mathbb{Z}^d}$ is given by the map $\sigma : \mathbb{Z}^d \times \Sigma^{\mathbb{Z}^d} \to \Sigma^{\mathbb{Z}^d}$ where

$$\sigma^u(x)_v := \sigma(u, x)_v = x_{u+v} \quad \text{for every } u, v \in \mathbb{Z}^d, x \in \Sigma^{\mathbb{Z}^d}.$$

Asymptotic pair

Two configurations $x, y \in \Sigma^{\mathbb{Z}^d}$ are **asymptotic** if they differ in finitely many sites of $\mathbb{Z}^d$.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$\sigma^{(4,1)}x$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

(both restricted to $[-5, 5] \times [-4, 3]$)

The set $F = \{\boldsymbol{n} \in \mathbb{Z}^d : x_{\boldsymbol{n}} \neq y_{\boldsymbol{n}}\}$ is called the **difference set** of (x, y).

Pattern, Cylinder, Language, Occurrences

- For finite subset $S \subset \mathbb{Z}^d$, an function $p \colon S \to \Sigma$ is called a **pattern** and the set S is its **support**. We denote it $p \in \Sigma^S$.
- Given a pattern $p \in \Sigma^S$, the **cylinder** centered at p is

$$[p] = \{x \in \Sigma^{\mathbb{Z}^d} : x|_S = p\}.$$

- For finite subset $S \subset \mathbb{Z}^d$, the **language with support S** of a configuration x is the set of patterns

$$\mathcal{L}_S(x) = \{p \in \Sigma^S : \text{ there is } \boldsymbol{n} \in \mathbb{Z}^d \text{ such that } \sigma^{\boldsymbol{n}}(x) \in [p]\}.$$

  The **language of x** is the union $\mathcal{L}(x)$ of the sets $\mathcal{L}_S(x)$ for every finite $S \subset \mathbb{Z}^d$.
- The **occurrences** of a pattern $p \in \Sigma^S$ in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ is

$$\mathrm{occ}_p(x) := \{\boldsymbol{n} \in \mathbb{Z}^d : \sigma^{\boldsymbol{n}}(x) \in [p]\}.$$

Language

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The **language** of patterns of support $S = \{\mathbf{0}, \boldsymbol{e}_1, 2\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in x is

$$\mathcal{L}_S(x) = \left\{ \begin{array}{llll} \begin{smallmatrix} 0 & & \\ 2 & 1 & 0 \end{smallmatrix}, & \begin{smallmatrix} 0 & & \\ 2 & 1 & 1 \end{smallmatrix}, & \begin{smallmatrix} 1 & & \\ 0 & 2 & 1 \end{smallmatrix}, & \begin{smallmatrix} 1 & & \\ 2 & 2 & 1 \end{smallmatrix}, \\ \begin{smallmatrix} 1 & & \\ 0 & 2 & 2 \end{smallmatrix}, & \begin{smallmatrix} 2 & & \\ 0 & 2 & 2 \end{smallmatrix}, & \begin{smallmatrix} 2 & & \\ 1 & 0 & 2 \end{smallmatrix}, & \begin{smallmatrix} 2 & & \\ 1 & 1 & 0 \end{smallmatrix} \end{array} \right\}$$

Also $\mathcal{L}_S(x, y) := \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$ for every finite support $S \subset \mathbb{Z}^d$.

Occurrences within asymptotic pairs

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The occurrences of the pattern $p =$

1		
0	2	1

in x and y are

$$\mathrm{occ}_p(x) = \{(-5, 2), (1, 1), (3, -3), \ldots\},$$
$$\mathrm{occ}_p(y) = \{(-5, 2), (1, 1), (3, -3), \ldots\},$$

Occurrences within asymptotic pairs

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

The occurrences of the pattern $p =$

1		
0	2	1

in x and y are

$$\begin{aligned} \mathrm{occ}_p(x) &= \{(-5,2),(1,1),(3,-3),\ldots\}, \\ \mathrm{occ}_p(y) &= \{(-5,2),(1,1),(3,-3),\ldots\}, \\ \mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y) &= \{(0,0)\}, \\ \mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x) &= \{(-2,-1)\}. \end{aligned}$$

Indistinguishable asymptotic pair

Let $p \in \Sigma^S$ is a pattern of finite support $S \subset \mathbb{Z}^d$.

If $x, y \in \Sigma^{\mathbb{Z}^d}$ are asymptotic configurations with difference set F, then

$$\mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y) = \mathrm{occ}_p(x) \cap (F - S)$$

and in particular it **is finite**.

Definition

We say that (x, y) is an **indistinguishable asymptotic pair** if (x, y) is asymptotic and the following equality holds

$$\#(\mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y)) = \#(\mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x))$$

for every pattern p of finite support.

Not all asymptotic pair is indistinguishable

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

The occurrences of the pattern $p =$

1	
	2

in x and y are

$$\begin{aligned} \mathrm{occ}_p(x) &= \{(-1,-1)\}, \\ \mathrm{occ}_p(y) &= \varnothing, \\ \mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y) &= \{(-1,-1)\}, \\ \mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x) &= \varnothing. \end{aligned}$$

Initial question

In Fall 2019, Sebastian Barbieri asked :

Question

Is there any non trivial pair $x, y \in \Sigma^{\mathbb{Z}^d}$ of indistinguishable asymptotic configurations ?

A trivial pair refers to cases like (x, x) and $(x, \sigma^{\boldsymbol{n}}(x))$ where $\boldsymbol{n} \in \mathbb{Z}^d$.

S. Barbieri, R. Gómez, B. Marcus, T. Meyerovitch, and S. Taati. Gibbsian representations of continuous specifications : the theorems of Kozlov and Sullivan revisited. Communications in Mathematical Physics, *382(2) :1111–1164, 2021.*

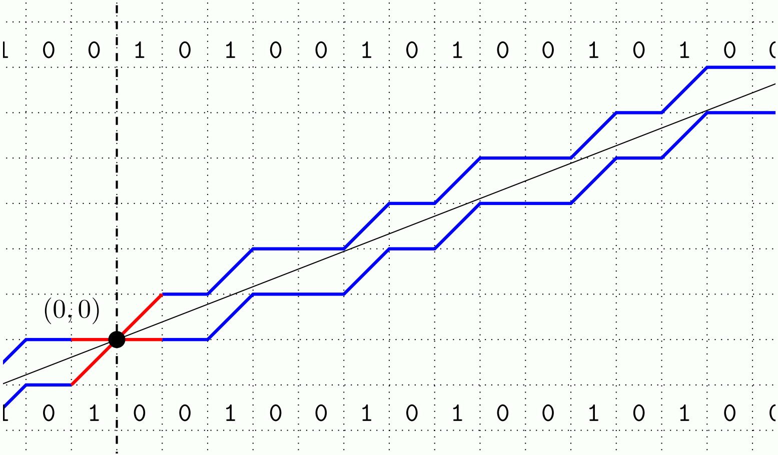
Outline

Sturmian words (Morse, Hedlund, 1940)

Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and $c_\alpha, c'_\alpha : \mathbb{Z} \to \{0, 1\}$ be the configurations

$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor$ (**lower** characteristic Sturmian word)

$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil$ (**upper** characteristic Sturmian word)



... are indistinguishable

$$c_\alpha = \cdots 101001010010\boxed{1.0}010010100101\cdots$$

$$c'_\alpha = \cdots 101001010010\boxed{0.1}010010100101\cdots$$

... are indistinguishable and vice versa

$$c_\alpha = \cdots 101001010010\boxed{1.0}010010100101\cdots$$
$$c'_\alpha = \cdots 101001010010\boxed{0.1}010010100101\cdots$$

Theorem (Barbieri, L, Starosta, 2021)

Let $x, y \in \{0,1\}^{\mathbb{Z}}$ and assume that x is ***recurrent****.*
The pair (x, y) is an ***indistinguishable asymptotic pair*** *with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$*

if and only if

there exists $\alpha \in [0,1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper ***characteristic Sturmian words*** *of slope α.*

Barbieri, L., Starosta, A characterization of Sturmian sequences by indistinguishable asymptotic pairs, European Journal of Combinatorics **95** (2021) 103318, doi:10.1016/j.ejc.2021.103318

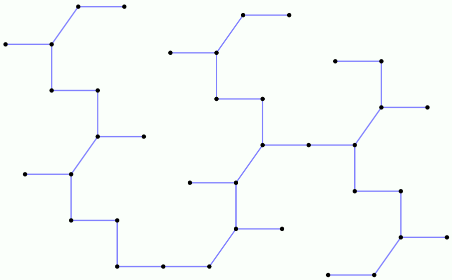
Outline

Discrete planes

A **discrete plane** of **normal vector** $v \in \mathbb{R}^3$, **intercept** μ and **width** ω is the subset

$$\{p \in \mathbb{Z}^3 \mid 0 \leq p \cdot v + \mu < \omega\} \subset \mathbb{Z}^3.$$

For example, with $\mu = 0$ and $\omega = \|v\|_1/2$, we get :



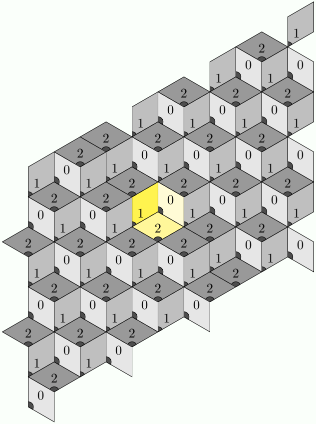
P. Arnoux, V. Berthé, and S. Ito. Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions. Annales de l'Institut Fourier, *52(2) :305–349, 2002.*

D. Jamet and J.-L. Toutant, On the connectedness of rational arithmetic discrete hyperplanes, LNCS 4245, 223–234, 2006.

Encoding upper and lower discrete planes

When $\mu = 0$ and $\omega = \|v\|_1$, we get the vertices of the surface of a **standard** discrete plane of normal vector $v \in \mathbb{R}^3$:

$\{p \in \mathbb{Z}^3 \mid 0 \leq p \cdot v < \|v\|_1\} \subset \mathbb{Z}^3$ Encoding : $\mathbb{Z}^2 \to \{0, 1, 2\}$



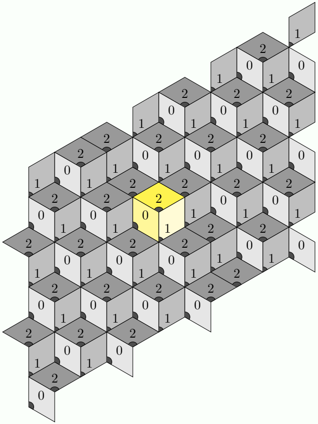
1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

Damien Jamet, Coding Stepped Planes and Surfaces by Two-Dimensional Sequences over a Three-Letter Alphabet 05047, 2005, pp.21

Encoding upper and lower discrete planes

When $\mu = 0$ and $\omega = \|v\|_1$, we get the vertices of the surface of a **standard** discrete plane of normal vector $v \in \mathbb{R}^3$:

$$\{p \in \mathbb{Z}^3 \mid 0 < p \cdot v \leq \|v\|_1\} \subset \mathbb{Z}^3 \qquad \text{Encoding} : \mathbb{Z}^2 \to \{0, 1, 2\}$$



1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

Damien Jamet, Coding Stepped Planes and Surfaces by Two-Dimensional Sequences over a Three-Letter Alphabet 05047, 2005, pp.21

d-dimensional Sturmian configurations

Let $\boldsymbol{\alpha} \in [0,1)^d$ be a **totally irrational vector**, that is,

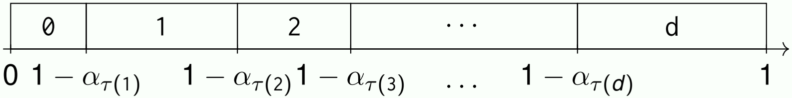
$$\boldsymbol{n} \in \mathbb{Z}^d, \boldsymbol{n} \cdot \boldsymbol{\alpha} \in \mathbb{Z} \implies \boldsymbol{n} = \boldsymbol{0}.$$

Definition

The **lower** and **upper characteristic d-dimensional Sturmian configurations** with slope $\boldsymbol{\alpha}$ are respectively :

$$\begin{array}{rccl} c_{\boldsymbol{\alpha}}: & \mathbb{Z}^d & \to & \{0,1,\ldots,d\} \\ & \boldsymbol{n} & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + \boldsymbol{n}\cdot\boldsymbol{\alpha} \rfloor - \lfloor \boldsymbol{n}\cdot\boldsymbol{\alpha} \rfloor\right) \end{array}$$

$$\begin{array}{rccl} c'_{\boldsymbol{\alpha}}: & \mathbb{Z}^d & \to & \{0,1,\ldots,d\} \\ & \boldsymbol{n} & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + \boldsymbol{n}\cdot\boldsymbol{\alpha} \rceil - \lceil \boldsymbol{n}\cdot\boldsymbol{\alpha} \rceil\right). \end{array}$$



Example

With $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (\sqrt{2}/2, \sqrt{19} - 4)$:

$c_{\boldsymbol{\alpha}} : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	1	0	2	1	0	2	1
0	2	1	0	2	2	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

$c'_{\boldsymbol{\alpha}} : \mathbb{Z}^2 \to \{0, 1, 2\}$

1	0	2	2	1	0	2	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0
2	1	0	2	2	1	0	2	1	0	2
1	0	2	1	0	2	2	1	0	2	1
0	2	1	0	2	1	1	0	2	1	0
2	1	0	2	1	0	2	2	1	0	2
1	0	2	1	0	2	1	1	0	2	1
0	2	1	0	2	1	0	2	2	1	0

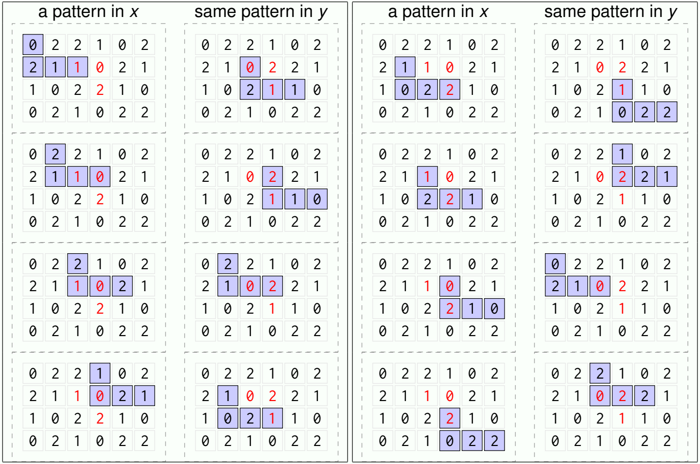
is an indistinguishable asymptotic pair encoding the projection of the surface of a discrete plane of normal vector

$$(1 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2) \approx (0.293, 0.348, 0.359).$$

Proposition (Barbieri, L., 2022)

Let $d \geq 1$. Let $\alpha \in [0,1)^d$ be a totally irrational vector. The lower and upper ***characteristic d-dimensional Sturmian configurations*** *(c_α, c'_α) with slope α is an* ***indistinguishable asymptotic pair****.*

The 8 patterns of support $\{\mathbf{0}, \boldsymbol{e}_1, 2\boldsymbol{e}_1, \boldsymbol{e}_2\}$ appearing in c_α and c'_α :



Proposition (Barbieri, L., 2022)

Let $d \geq 1$. Let $\alpha \in [0,1)^d$ be a totally irrational vector. The lower and upper ***characteristic d-dimensional Sturmian configurations*** *(c_α, c'_α) with slope α is an* ***indistinguishable asymptotic pair****.*

Question

What about the reciprocal ?

Flip condition

Definition

An asymptotic pair $x, y \in \{0, 1, \dots, d\}^{\mathbb{Z}^d}$ satisfies the **flip condition** if

1. the difference set of x and y is $F = \{\mathbf{0}, -\mathbf{e}_1, \dots, -\mathbf{e}_d\}$,
2. the restriction $x|_F$ is a **bijection** $F \to \{0, 1, \dots, d\}$,
3. the map defined by $x_{\mathbf{n}} \mapsto y_{\mathbf{n}}$ for every $\mathbf{n} \in F$ is a **cyclic permutation** on the alphabet $\{0, 1, \dots, d\}$.

Without lost of generality, we assume that $x_{\mathbf{0}} = 0$ and $y_{\mathbf{n}} = x_{\mathbf{n}} - 1 \mod (d+1)$ for every $\mathbf{n} \in F$.

$x : \mathbb{Z}^2 \to \{0, 1, 2\}$

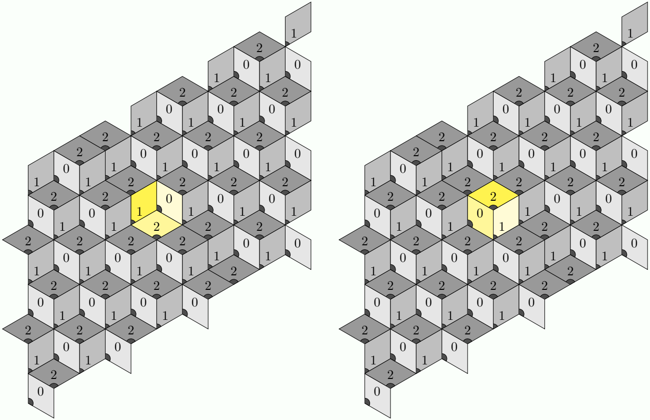
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$y : \mathbb{Z}^2 \to \{0, 1, 2\}$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Flip condition

The flip condition may be interpreted as the **geometrical flip** of the faces of a hypercube at the origin of a **discrete hyperplane** :



T. Jolivet. Combinatorics of Pisot Substitutions. *PhD Thesis, 2013.*

Damien Jamet, Coding Stepped Planes and Surfaces by Two-Dimensional Sequences over a Three-Letter Alphabet 05047, 2005, pp.21

Theorem B

Theorem B (Barbieri, L., 2022)

Let $d \geq 1$ and $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ s.t. x is **uniformly recurrent**. The pair (x, y) is an **indistinguishable asymptotic pair** satisfying the flip condition

if and only if

there exists a **totally irrational vector** $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper **characteristic d-dimensional Sturmian configurations** with slope α.

(Theorem B depends on Theorem A)

Theorem A

Theorem A (Barbieri, L., 2022)

Let $d \geq 1$ and $x, y \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be an asymptotic pair satisfying the **flip condition** with difference set $F = \{\mathbf{0}, -\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_d\}$. The following are equivalent :

(i) For every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, we have

$$\#\left(\mathrm{occ}_p(x) \setminus \mathrm{occ}_p(y)\right) = 1 = \#\left(\mathrm{occ}_p(y) \setminus \mathrm{occ}_p(x)\right).$$

(ii) The asymptotic pair (x, y) is **indistinguishable**.

(iii) For every nonempty finite **connected** subset $S \subset \mathbb{Z}^d$, the **pattern complexity** of x and y is

$$\#\mathcal{L}_S(x) = \#\mathcal{L}_S(y) = \#(F - S).$$

Complexity $\#(F - S)$

Complexity $\#(F - S)$ matches what is known :

- When $d = 1$ and $S = \{0, 1, \dots, n-1\}$:

$$\#(F - S) = \#\left(\{0, -1\} - \{0, 1, \dots, n-1\}\right) = n + 1$$

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- When $d = 2$ and $S = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$:,

$$\begin{aligned}\#(F - S) &= \#\left(\{\mathbf{0}, -\boldsymbol{e}_1, -\boldsymbol{e}_2\} - \{(i,j)\colon 0 \le i < n, 0 \le j < m\}\right)\\ &= mn + m + n\end{aligned}$$

  is the rectangular pattern complexity of a **discrete plane** with totally irrational slope.

V. Berthé, L. Vuillon. Tilings and rotations on the torus : a two-dimensional generalization of Sturmian sequences. Discrete Mathematics, *223(1-3) :27–53, 2000.*

Outline

Open question 1

Theorem (Barbieri, L, Starosta, 2021)

Let $x, y \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}$.
The pair (x, y) is an ***indistinguishable asymptotic pair*** *with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = \mathtt{10}$ and $y_{-1}y_0 = \mathtt{01}$*

if and only if

there exists a monotone sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in [0,1] \setminus \mathbb{Q}$ s.t. $x = \lim_{n\to\infty} c_{\alpha_n}$ and $y = \lim_{n\to\infty} c'_{\alpha_n}$ are the limits of ***characteristic Sturmian words*** *of slope α_n.*

Moreover, indistinguishable asymptotic pairs over $\mathbb{Z}$ for any finite difference set F are described in terms of derived sequences.

Open question

Describe indistinguishable asymptotic pair $x, y \in \Sigma^{\mathbb{Z}^d}$ satisfying the flip condition where x is **not uniformly recurrent**.

Open question 2

A sequence $w \in \Sigma^{\mathbb{Z}}$ with $\#\mathcal{L}_n(w) \leq n$ is eventually periodic.

Nivat's conjecture

A configuration $x \in \Sigma^{\mathbb{Z}^2}$ for which there are $n, m \in \mathbb{N}$ with $\#\mathcal{L}_{(n,m)}(x) \leq nm$ is periodic.

Equivalently, a sequence $w \in \Sigma^{\mathbb{Z}}$ with totally irrational vector of symbol frequencies has complexity $\#\mathcal{L}_n(w) \geq n + 1$.

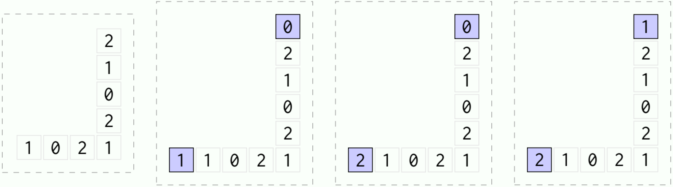
Dual Nivat Conjecture

Let $d \geq 1$ and $F = \{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_d\}$. Let $x \in \{0, 1, \ldots, d\}^{\mathbb{Z}^d}$ be a configuration with trivial stabilizer, i.e., $\sigma^{\mathbf{n}}(x) = x$ only holds for $\mathbf{n} = \mathbf{0}$. If the **frequencies** of symbols in x **exist and are rationally independent**, then $\#\mathcal{L}_S(x) \geq \#(F - S)$ for every nonempty connected finite support $S \subset \mathbb{Z}^d$.

J. Cassaigne. Double sequences with complexity $mn + 1$. volume 4, pages 153–170. 1999. Journées Montoises d'Informatique Théorique (Mons, 1998).

Open question 3

The pattern below is bispecial within the language of c_α and c'_α :



Bispecial factors within the language of a Sturmian sequence of slope $\alpha \in [0, 1)$ are related to the convergents of the continued fraction expansion of α (de Luca, 1997).

Question

Let $d \geq 1$ and $\boldsymbol{\alpha} \in [0, 1)^d$ be a totally irrational vector. What is the relation between the set

$$V_{\boldsymbol{\alpha}} = \left\{ b - a \colon \exists w \in \mathcal{L}(c_{\boldsymbol{\alpha}}) \text{ bispecial at positions } a, b \in \mathbb{Z}^d \right\}$$

and **simultaneous Diophantine approximations** of $\boldsymbol{\alpha}$?