The Penrose Tiling is a Quantum Error-Correcting code (QECC)

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Li → Quantum

Boyle → interests: cosmology, blackholes, Bigbang, neutrino darkmatter, Penrose tilings

① Linear code (Wikipedia)

Def A linear code of length n and dimension k is a linear subspace C with dimension k of the vector space $\mathbb{F}_q^n$ (binary when $K=2$)

EX $[7,4,3]=[n,k,d]$ Hamming code represents 4-bit messages using 7bit code words, where two distinct code words differ in at least 3 bits (Hamming distance).

② QECC

We want to protect a state $|\psi\rangle$ in a Hilbertspace $\mathcal{H}_0$ (∞-dim. vector space)

$\mathcal{H}_0$ is ~~embedded~~ in an enlarged Hilbertspace $\mathcal{H}$ as a carefully chosen subspace $\mathcal{C}_a$, called code space.

Theorem (Knill, Laflamme, 1996)

The code $\mathcal{C}_a$ ~~can~~ be extended to an $\mathcal{A}$-correcting code iff for all basis elements $|\psi_i\rangle, |\psi_j\rangle$ and operators $A_a, A_b \in \mathcal{A}$

$$\langle \psi_i | A_a^\dagger A_b | \psi_i \rangle = \langle \psi_j | A_a^\dagger A_b | \psi_j \rangle, \forall i,j$$

and

$$\langle \psi_i | A_a^\dagger A_b | \psi_j \rangle = 0 \text{ si } i \neq j.$$

~~2 Equivalent Reformulations (1. Quantum recoverability)~~ ~~"partial traces"~~

~~$\exists$ quantum channel R st~~

~~$\forall |\xi\rangle \in \mathcal{C}_a, R(tr_K |\xi\rangle\langle\xi|) = |\xi\rangle\langle\xi|$~~

~~2. Quantum indistinguishability~~ ~~(†)~~

~~$Tr_{K^c} |\xi\rangle\langle\xi|$ is ind. of $|\xi\rangle \in \mathcal{C}$~~

~~K is a region where there are errors to be corrected~~

Error to be corrected: erasure of an arbitrary finite spatial region K. We write $\mathcal{H} = \mathcal{H}_K \otimes \mathcal{H}_{K^c}$

Two equivalent reformulations: "partial traces"

1. Quantum recoverability

$\exists$ quantum channel R s.t.

$$R(\mathrm{tr}_K |\xi\rangle\langle\xi|) = |\xi\rangle\langle\xi| \quad \forall |\xi\rangle \in \mathcal{C}$$

2. Quantum indistinguishability

$\mathrm{tr}_{K^c} |\xi\rangle\langle\xi|$ is ind. of $|\xi\rangle \in \mathcal{C}$

$$\Leftrightarrow \mathrm{tr}_{K^c} |\psi_i\rangle\langle\psi_j| = \langle\psi_j|\psi_i\rangle \rho_K \quad \forall i,j$$

where $\mathcal{C}$ is span by states $|\psi_i\rangle$.

(3) Penrose tilings and Jeandel-Rao tilings

Def (Baake, Grimm) Two tilings T, T' of $\mathbb{R}^d$ are locally indistinguishable, $T \overset{LI}{\sim} T'$, when any pattern in T occurs in T' and vice versa.

EX If T, T' are two Penrose tilings then $T \overset{LI}{\sim} T'$.

Def A stronger quantitative version of local indistinguibility is when also the relative frequencies of different finite patches are also the same.

Def (local recoverability) when the pattern in any finite region K can be uniquely recovered from the pattern in the complementary region K^c.

EX (with Jeandel-Rao) Complete the empty region.

10	2	8
8		3
1	1	1

6	6	7	4
7			10
3			8
3	9	9	9

10	2	8	7	3
8				9
1				0
6				4
5	7	4	3	10

6	6	6	7	4	5
5					5
7	?	?	?		7
9					9
0					0
7	5	7	4	5	7

④ QECC from Penrose

Let T be a Penrose tiling.

$[T] = \{ gT \mid g \text{ is orientation preserving isometry} \}$

"We can regard a tiling as a state $|T\rangle$ in a quantum mechanical Hilbert space $\mathcal{H}$."

$\mathbb{R}^2 = K \dot\cup K^c$, K finite region

$T = T_K \dot\cup T_{K^c}$

$\mathcal{H} = \mathcal{H}_K \otimes \mathcal{H}_{K^c}$, decomposition of Hilbert space

$|T\rangle = |T\rangle_K |T\rangle_{K^c}$, $|T\rangle_K \in \mathcal{H}_K$, $|T\rangle_{K^c} \in \mathcal{H}_{K^c}$

Distinct states are orthogonal:

$$\langle T', T\rangle = \delta(T', T)$$

to each equivalence class $[T]$, define the wavefunction

$$|\Psi_{[T]}\rangle = \int dg\, |gT\rangle$$

The main claim of this paper: states $|\Psi_{[T]}\rangle$ form an orthogonal basis for the code space $C < \mathcal{H}$ of a QECC that corrects arbitrary errors or erasures in any finite region K.

Proof (based on Penrose tilings recoverability and local indistinguability)

Let T, T' two Penrose tilings, K some finite region.

We want to check $\operatorname{tr}_{K^c} |\Psi_{[T]}\rangle\langle\Psi_{[T']}| = \langle \Psi_{[T]}, \Psi_{[T']}\rangle P_K$.

Suppose $[T] \neq [T']$. By PT recoverability, they differ on K^c, (logical problem in the paper here $p \Rightarrow q$ vs $q \Rightarrow p$ or just sloppy) thus left part $= 0$.

Also, $[T] \neq [T'] \Rightarrow \langle T', T\rangle = 0 \Rightarrow \langle \Psi_{[T]}, \Psi_{[T']}\rangle = 0$.

Suppose $[T] = [T']$. We get

$$\operatorname{Tr}_{K^c} |\Psi_{[T]}\rangle\langle\Psi_{[T]}| \overset{[\dots]}{=} \underbrace{\int dg'' \delta(T, g''T)}_{\substack{\text{proportional to} \\ \langle \Psi_{[T]} | \Psi_{[T]}\rangle}} \cdot \underbrace{\int dg\, |gT\rangle_K \langle gT|_K}_{= P_K}$$

$= P_K$ uses strong version local-indistinguability

"does not depend on T" ← uniform pattern frequency □