

A q -analogue of Markoff injectivity
conjecture is true

joint work with Mélodie Lapointe
and Wolfgang Steiner

Sébastien Labbé
CNRS, LaBRI, Université de Bordeaux

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Dyadisc 7

Dyadisc 7: Brazilian-chilean and french interplay for symbolic dynamics.

<https://dyadisc7.sciencesconf.org/>

References

Sur les formes quadratiques binaires indéfinies.

Von

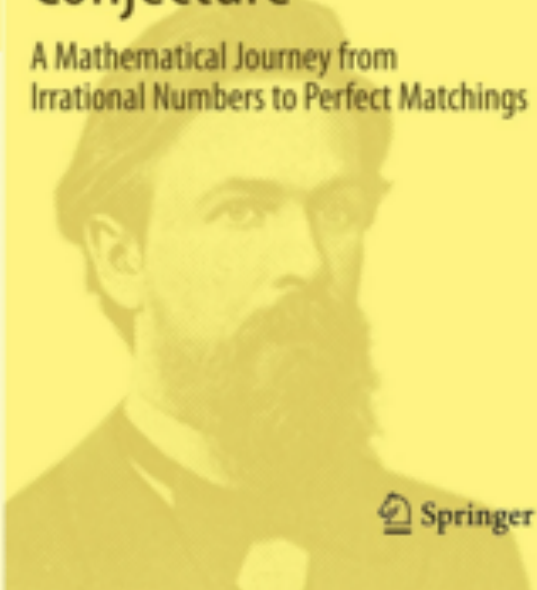
A. MARKOFF in St. Petersburg.

Markoff
(1879, 1880)

Martin Aigner

Markov's Theorem and 100 Years of the Uniqueness Conjecture

A Mathematical Journey from
Irrational Numbers to Perfect Matchings



Le mémoire „Sur les formes quadratiques“ de M. M. A. K. (Markoff*) a été mentionné que la limite précise de l'ensemble des formes binaires

$$f = ax^2 + 2bxy + cy^2,$$

Le discriminant $b^2 - ac = D$ est positif, est égal à $\sqrt{\frac{4}{5}D}$ et le minimum des formes équivalentes à

$$f_0 = \sqrt{\frac{4}{5}D} (x^2 - xy - y^2);$$

et les autres formes f la limite précise de leurs minimums est $\sqrt{\frac{1}{2}D}$.

La démonstration de ces théorèmes m'étant communiquée par M. Markoff, ainsi que la forme

$$f_1 = \sqrt{\frac{1}{2}D} (x^2 - 2xy - y^2)$$

qui sont équivalentes ont $\sqrt{\frac{1}{2}D}$ pour leur minimum, je me suis efforcé de trouver la quantité qui doit remplacer $\sqrt{\frac{1}{2}D}$ pour les formes non équivalentes à f_0 et f_1 . Il résulte de ce que cette quantité $\sqrt{\frac{100}{221}D}$ est le minimum des formes

$$f_2 = \sqrt{\frac{4D}{221}} (5x^2 - 11xy - 5y^2).$$

Pour ne pas nous occuper des cas particuliers, abordons les questions générales et proposons nous de trouver les formes f , dont les valeurs ne puissent être inférieures à $l\sqrt{D}$. Nous allons démontrer, que pour

*) Mathematische Annalen, Band VI, S. 366.

From CHRISTOFFEL WORDS to MARKOFF NUMBERS

Christophe Reutenauer

OXFORD

Aigner
(2013)

Reutenauer
(2018)

Markoff numbers

Def A Markoff triplet is a positive integer solution to the equation

$$x^2 + y^2 + z^2 = 3xyz$$

EX: $(1, 1, 1), (1, 1, 2), (1, 2, 5), \dots$

Def An integer is a Markoff number if it appears in a Markoff triplet.

EX: $1, 2, 5, \dots$ are Markoff numbers

Lemma If (x, y, z) is a M. triplet, then $(3yz-x, y, z)$ also.

Proof $(3yz-x)^2 + y^2 + z^2 = 9y^2z^2 - 6xyz + x^2 + y^2 + z^2$
 $= 9y^2z^2 - 3xyz = 3(3yz-x)yz.$

Also, $3yz-x = \frac{y^2+z^2}{x} > 0. \square$

Assuming $x < y < z$, then we have 3 other triplets:

$(3xy-z, x, y)$ with $3xy-z < y$

Ref: Markoff, 1880
 p. 397
 or Reutenauer
 Lemma 31.6

↓

$(x < y < z)$

↙

$(x < z < \underline{3xz-y})$

↘

$(y < z < \underline{3yz-x})$

the maximum of
 a triplet is
underlined

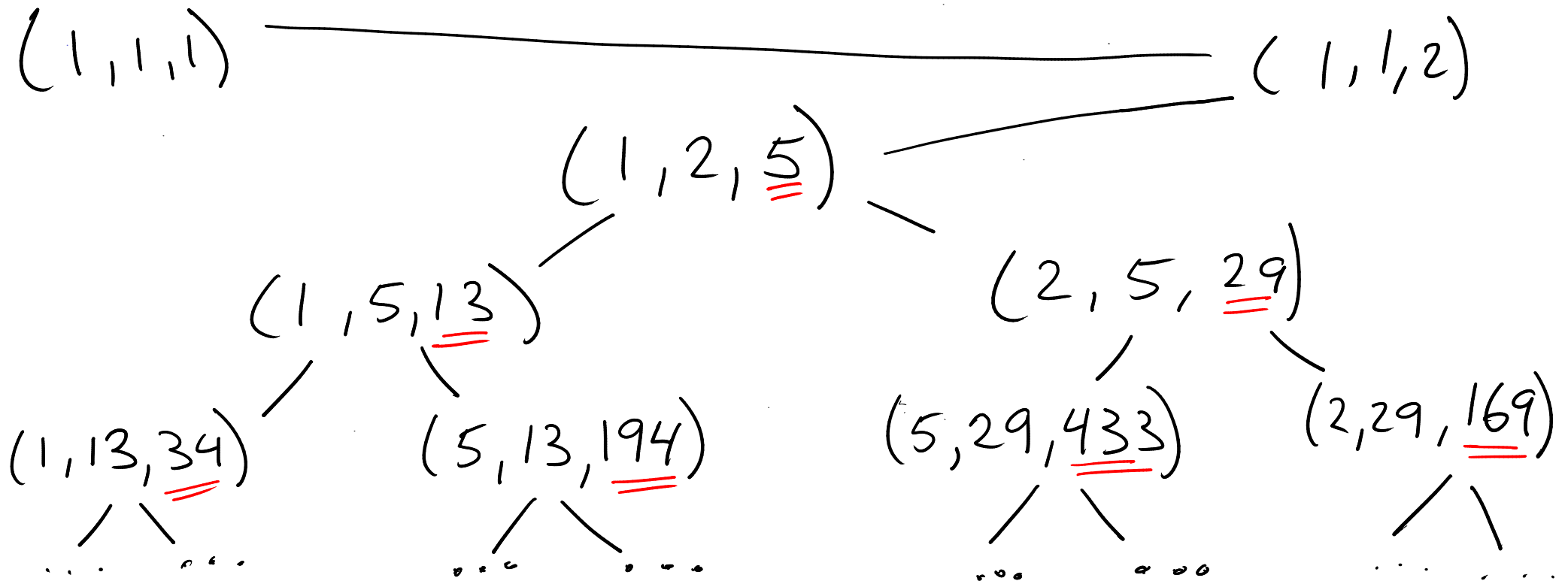
Because

$3xz-y > 3xz-z = (3x-1)z > z$

Because $3yz-x > 3yz-z = (3y-1)z > z.$

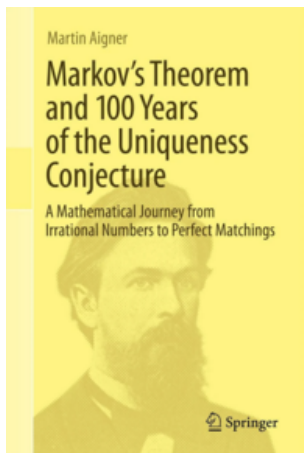
The tree of Markoff triplets

Proposition (Markoff, 1880) Every Markoff triplet appears in the tree



The Uniqueness Conjecture (Frobenius, 1913)

Every Markoff number is the maximum of a unique Markoff triplet.



Motivation for the Conjecture (Dio. Approximations)

Theorem Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $\exists \infty$ $P/Q \in \mathbb{Q}$ distinct

s.t. $|\alpha - \frac{P}{Q}| < \begin{cases} \frac{1}{Q^2} & \text{(Dirichlet, 1842)} \\ \frac{1}{\sqrt{5}Q^2} & \text{(Hurwitz, 1891)} \\ \frac{1}{\sqrt{8}Q^2} & \text{if } \alpha = [a_0; a_1, a_2, a_3, \dots] \\ & \text{with } \#\{n: a_n \geq 2\} = \infty \\ \dots & \dots \end{cases}$

Lagrange Spectrum

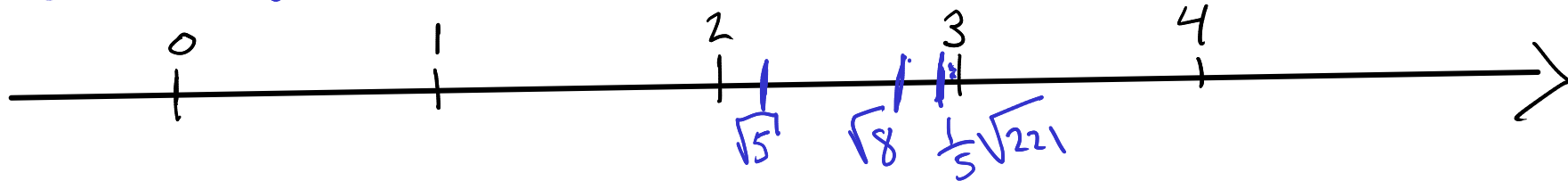
$$L(\alpha) = \sup \left\{ L : |\alpha - \frac{P}{Q}| < \frac{1}{LQ^2} \text{ for } \infty\text{-many } P/Q \in \mathbb{Q} \right\}$$

$$\mathcal{L} = \{L(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$$

Rmk. $\alpha \in \mathbb{Q} \Leftrightarrow L(\alpha) = 0$
 $\alpha \notin \mathbb{Q} \Leftrightarrow L(\alpha) \geq \sqrt{5}$

Theorem (Markoff, 1879, 1880)

$$\mathcal{L} \cap (-\infty, 3) = \left\{ \sqrt{9 - \frac{4}{m^2}} : m \text{ is a Markoff number} \right\}$$



$$L\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5} = L\left(\frac{3+\sqrt{5}}{2}\right) = \dots$$

$$L(\sqrt{2}+1) = \sqrt{8} = L(2\sqrt{2}+3) = \dots$$

Definition $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ We write $\alpha \sim \beta$ if their continued fraction expansion eventually coincide

i.e. $\alpha = [a_0; a_1, a_2, \dots, a_k, \delta]$ and

$\beta = [b_0; b_1, b_2, \dots, b_\ell, \delta]$ for some $\delta \in (\mathbb{N}_{>0})^{\mathbb{N}}$.

Conjecture Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ with $L(\alpha), L(\beta) < 3$

$$\alpha \sim \beta \Leftrightarrow L(\alpha) = L(\beta).$$

Note: This conjecture is equivalent to the uniqueness conjecture.

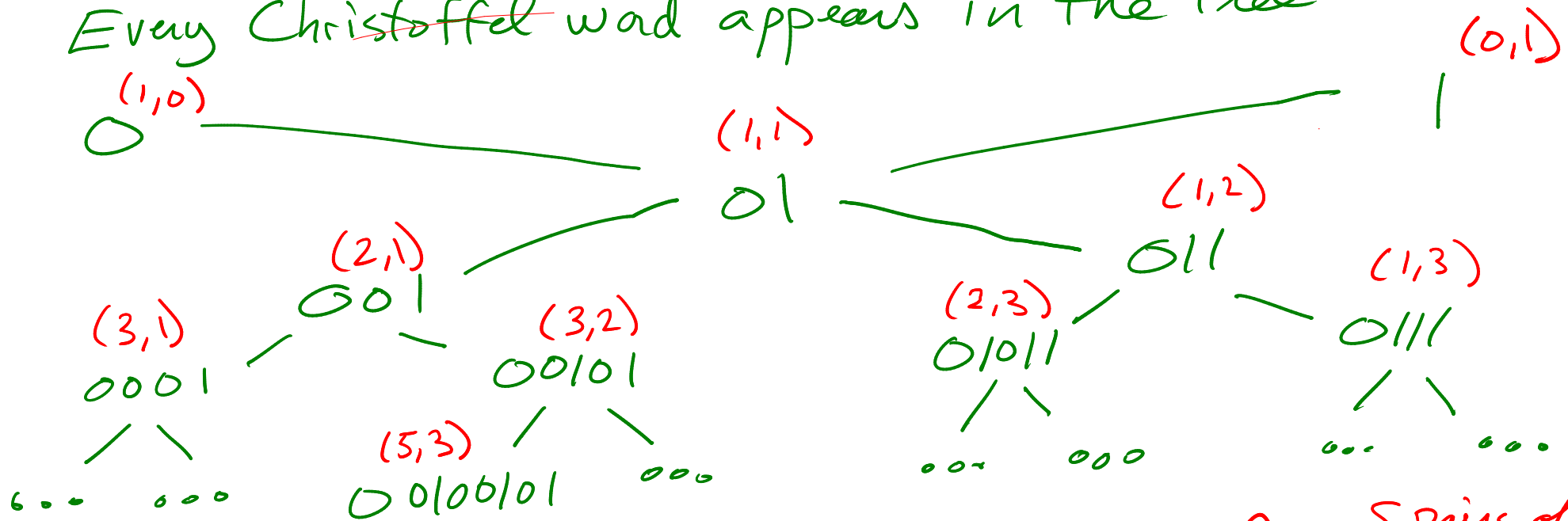
Def The set of Christoffel words is the smallest set $C \subset \{0,1\}^*$ s.t. $0 \in C, 1 \in C, 01 \in C$ and if $u, v, uv \in C$, then $uv, uvv \in C$.

EX:

- $0, 1, 01 \in C \Rightarrow 001, 011 \in C$
- $0, 01, 001 \in C \Rightarrow 0001, 00101 \in C$
- $0, 1, 011 \in C \Rightarrow 01011, 0111 \in C$

The tree of Christoffel words

Every Christoffel word appears in the tree



The map $w \mapsto (|w|_0, |w|_1)$ is a bijection $C \rightarrow \left\{ \begin{array}{l} \text{pairs of} \\ \text{coprime} \\ \text{integers} \end{array} \right\}$

Markoff injectivity conjecture

Let $\mu: \{0,1\}^* \rightarrow SL_2(\mathbb{Z})$ be the homomorphism defined by $\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$.

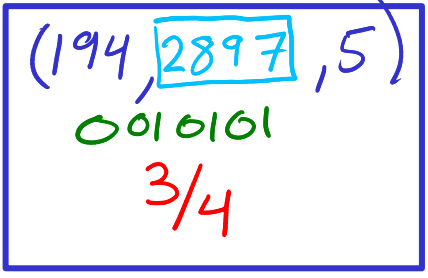
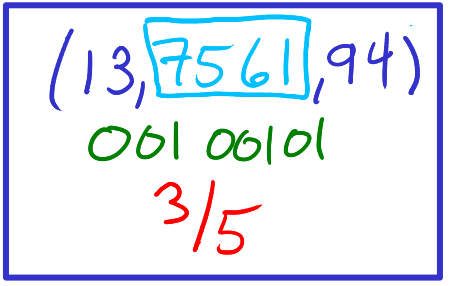
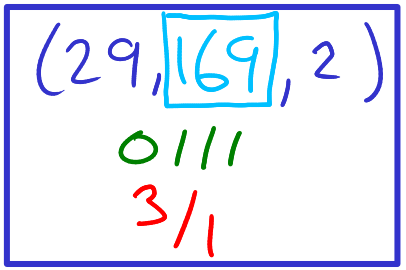
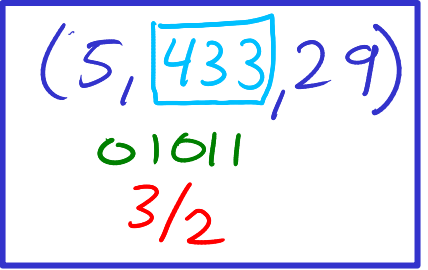
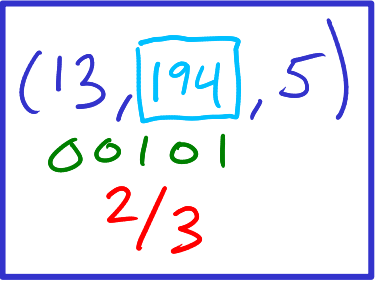
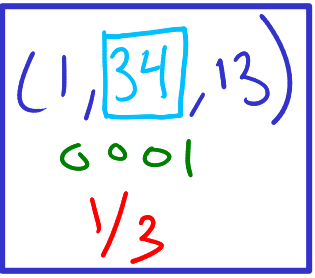
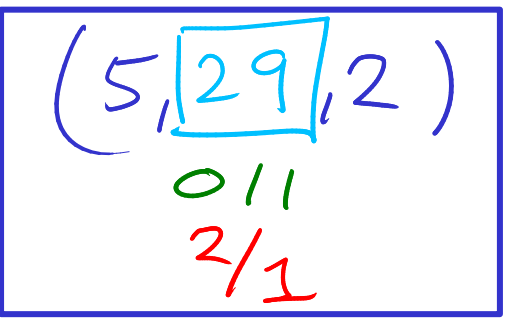
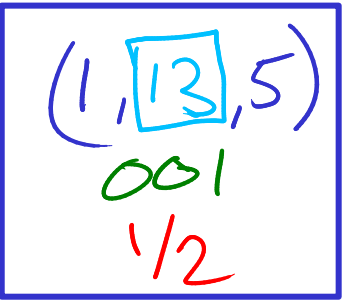
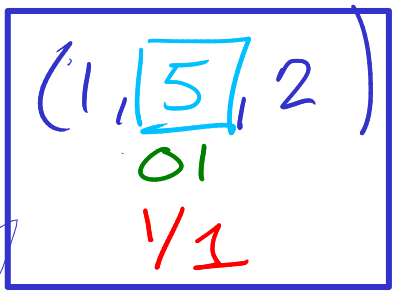
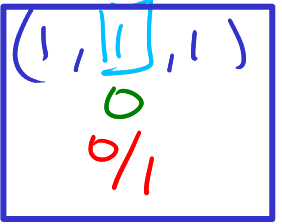
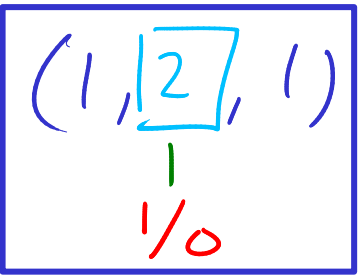
Reutenauer (2009) For every Markoff number m , there exists a Christoffel word $w \in C$ such that $m = \mu(w)_{12} := (1 \ 0) \mu(w) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

EX! $\mu(00101) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix}$

Markoff Injectivity Conjecture (\equiv the uniqueness conjecture)

The map $w \mapsto \mu(w)_{12}$ is injective over the set C of Christoffel words.

The three trees together



... 000

... 000

... 000

... 000

... 000

q-analogs

A q-analog of Something is Something_q such that

$$\lim_{q \rightarrow 1} \text{Something}_q = \text{Something}.$$

EX The q-analog of an integer $n \in \mathbb{N}$ is

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\dots+q^{n-1} \quad \text{and} \quad \lim_{q \rightarrow 1} [n]_q = n$$

EX The q-analog of the factorial $n!$ is

$$[n]_q! = [1]_q [2]_q \dots [n]_q \quad \text{and} \quad \lim_{q \rightarrow 1} [n]_q! = n!$$

We have

$$n! = \# S_n = \# \{ \text{permutations of } n \text{ elements} \} = \sum_{\sigma \in S_n} 1$$

$$[n]_q! = \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)}$$

where $\text{inv}(\sigma) = \# \text{inversions in the permutation } \sigma$

q -analogues of rational numbers

For every $\frac{r}{s} \in \mathbb{Q}_{>0}$, $\exists!$ integers $a_0 \geq 0$, $a_i > 0$, n odd s.t.

$$\frac{r}{s} = a_0 + \frac{1}{\frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_n}}}} = [a_0; a_1, a_2, \dots, a_n] \quad (n \text{ odd})$$

Also,

$$\begin{pmatrix} r \\ s \end{pmatrix} = R^{a_0} L^{a_1} \dots R^{a_{n-1}} L^{a_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where } R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(Morier-Genoud, Ovsienko 2020) Proposed:

$$\begin{pmatrix} R(q) \\ S(q) \end{pmatrix} = q^{-1} R_q^{a_0} L_q^{a_1} \dots R_q^{a_{n-1}} L_q^{a_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where } R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, L_q = \begin{pmatrix} q & 0 \\ 1 & 1 \end{pmatrix}$$

$$\text{and } \left[\frac{r}{s} \right]_q = \frac{R(q)}{S(q)}.$$

$$\underline{\text{EX}} \quad \left[\frac{2}{7} \right]_q = \frac{q^4 + q^3}{q^4 + 2q^3 + 2q^2 + q + 1}$$

$$\lim_{q \rightarrow 1} \left[\frac{2}{7} \right]_q = \frac{2}{7}.$$

A q -analogue of Markoff numbers

(Kogiso, 2020)

We have $\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = RL$

and $\mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = RRL$ with $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Let $L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$ and $R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$ where q is some indeterminate.

Let $\mu_q(0) = R_q L_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}$

$\mu_q(1) = R_q R_q L_q L_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$

which extends to a homomorphism $\mu_q: \{0,1\}^* \rightarrow GL_2(\mathbb{Z}[q, q^{-1}])$

Def: If $w \in \mathbb{C}$ is a Christoffel word, then $\mu_q(w)_{12}$ a q -analogue of a Markoff number.

EX: $\mu_q(00101)_{12} = 1 + 4q + 10q^2 + 18q^3 + 27q^4 + 33q^5 + 33q^6 + 29q^7 + 21q^8 + 12q^9 + 5q^{10} + q^{11}$

which evaluates at 194 when $q=1$.

Let $(\{0,1\}^*, <_{\text{radix}})$ be a total order s.t.

$$u <_{\text{radix}} v \quad \text{if} \quad \begin{cases} |u| < |v| \\ |u| = |v| \text{ and } u <_{\text{lex}} v \end{cases}$$

Let $(\mathbb{Z}[q], <)$ be a partial order on polynomials s.t.

$$f < g \Leftrightarrow f \neq g \text{ and } g - f \in \mathbb{Z}_{>0}[q]$$

Theorem (L, Lapointe, 2022)

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence.

For every $u, v \in \mathcal{L}(s)$

$$u <_{\text{radix}} v \Rightarrow \mu_q(u)_{12} < \mu_q(v)_{12}.$$

Corollary $w \mapsto \mu_q(w)_{12}$ is injective over the language $\mathcal{L}(s)$ of a balanced sequence $s \in \{0,1\}^{\mathbb{Z}}$.

Corollary (Lapointe, Rentenauer, 2021) $w \mapsto \mu(w)_{12}$ is injective the language of a balanced sequence.

Theorem (L, Lapointe, Steiner, 2023)

The map $w \mapsto \mu_g(w)_{12}$ is injective over the set C of Christoffel words.

Note: this is of course weaker than the Markoff injectivity conjecture because distinct polynomials can evaluate at the same value at $g=1$.

Idea of the proof: Evaluating the polynomials $\mu_g(w)_{12}$ at the sixth root of unity allows to recover the number of occurrences of 0 and 1 in w , thus the rational $\frac{|w|_1}{|w|_0}$. Injectivity follows from the isomorphism between the Christoffel tree and the Stern-Brocot tree, because no rational number appears twice in the Stern-Brocot tree.

Idea of the proof (not mentioned in the published version)

Let $z = e^{i\pi/3}$ root of $x^2 - x + 1$. Then

$$A = \mu_z(0)_{1,2} |_{q=z} = \begin{pmatrix} 2z^{-1} & 1 \\ z & 1 \end{pmatrix}$$

$$B = \mu_z(1)_{1,2} |_{q=z} = \begin{pmatrix} 2z-3 & z+1 \\ 2z-1 & 1 \end{pmatrix}$$

and

$$AB = \begin{pmatrix} -2z-2 & 3z-2 \\ z-3 & 2z \end{pmatrix} = BA.$$

Thus

$$\mu_q(w) = A^{|w|_0} B^{|w|_1}. \quad \text{Also } \langle A, B \rangle \cong \mathbb{Z}^2$$

Remark The map $w \mapsto \mu_g(w)_{12}$ is not injective over $\{0,1\}^*$.

Ex
$$\begin{aligned} \mu_g(00011)_{12} &= 1 + 4g + 10g^2 + 19g^3 + 27g^4 + 33g^5 + 34g^6 \\ &\quad + 29g^7 + 21g^8 + 12g^9 + 5g^{10} + g^{11} \\ &= \mu_g(01001)_{12} \end{aligned}$$

In general, we have

Lemma (Lapointe, Steiner) $\forall w \in \{0,1\}^*$ $\mu_g(0w1)_{12} = \mu_g(0\tilde{w}1)_{12}$

where $\tilde{w} = w_n \dots w_2 w_1$ denotes the reversal of $w = w_1 w_2 \dots w_n$.

Amnesty International Report, December 5th, 2024

#stopgaza genocide

1. EXECUTIVE SUMMARY

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KILLINGS AND SERIOUS INJURIES
INFLECTING CONDITIONS OF LIFE
SPECIFIC INTENT
CONCLUSION AND RECOMMENDATIONS

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**'YOU FEEL LIKE
YOU ARE SUBHUMAN'**

ISRAEL'S GENOCIDE AGAINST PALESTINIANS IN GAZA

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