A q-analogue of Markoff injectivity conjecture is true

joint work with Mélodie Lapointe and Wolfgang Steiner

Sébastien Labbé
CNRS, LaBRI, Université de Bordeaux

Amiens, December 11th, 2024
Dyadisc 7

Dyadisc 7: Brazilian–chilean and french interplay for symbolic dynamics.

https://dyadisc7.sciencesconf.org/

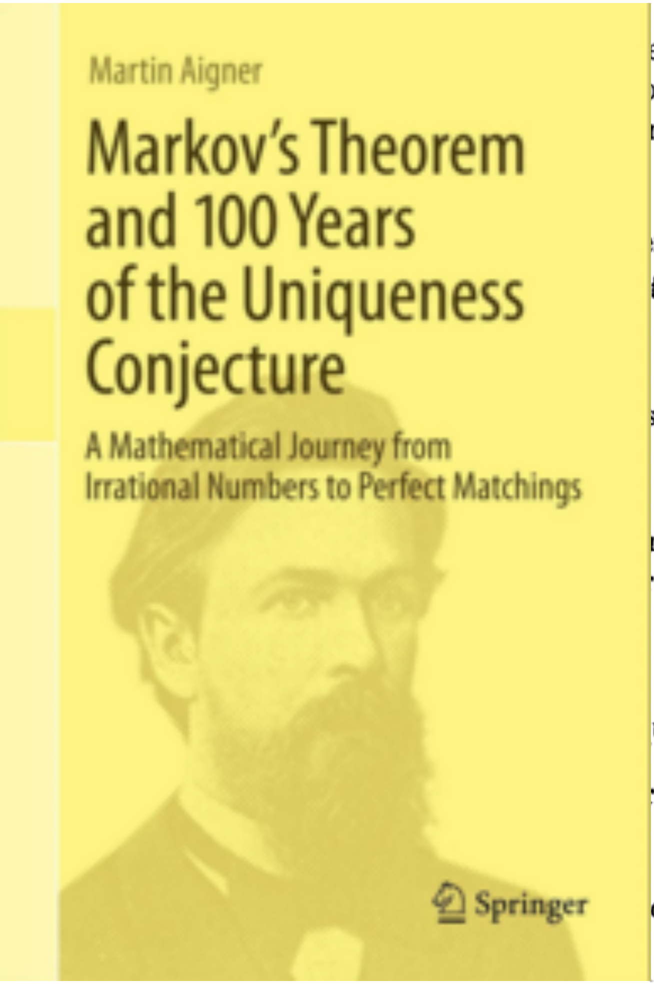
References

Markoff (1879, 1880)

Sur les formes quadratiques binaires indéfinies.

Von

A. Markoff in St. Petersburg.



e mémoire „Sur les formes quadratiques" de M. M. A. K
otareff*) a été mentionné que la limite précise de
mble des formes binaires

$$f = ax^2 + 2bxy + cy^2,$$

rminant $b^2 - ac = D$ est positif, est égal à $\sqrt{\frac{4}{5}}$
t le minimum des formes équivalentes à

$$f_0 = \sqrt{\tfrac{4}{5} D}\,(x^2 - xy - y^2);$$

s les autres formes f la limite précise de leurs mi

$$\sqrt{\tfrac{1}{2} D}.$$

monstration de ces théorèmes m'étant communi
kine, ainsi que la forme

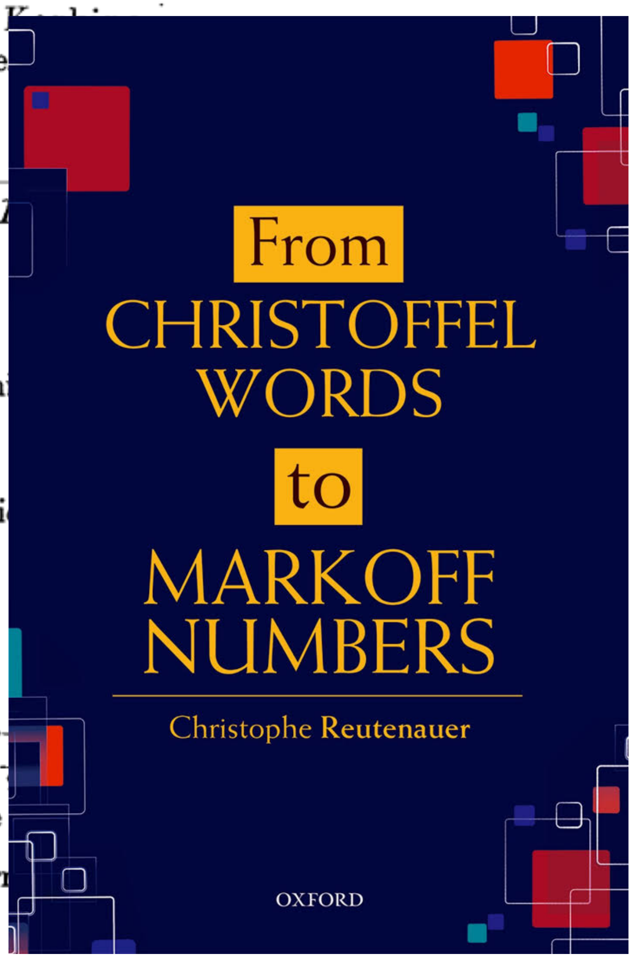
$$f_1 = \sqrt{\tfrac{1}{2} D}\,(x^2 - 2xy - y^2)$$

uivalentes ont $\sqrt{\frac{1}{2} D}$ pour leur minimum, je me
d de trouver la quantité qui doit remplacer $\sqrt{}$
formes non équivalentes à f_0 et f_1. Il résulte
que cette quantité $\sqrt{\frac{100}{221} D}$ est le minimum des for

$$f_2 = \sqrt{\tfrac{4D}{221}}\,(5x^2 - 11xy - 5y^2).$$

Pour ne pas nous occuper des cas particuliers, abordons les questions générales et proposons nous de trouver les formes f, dont les valeurs ne puissent être inférieures à $l\sqrt{D}$. Nous allons démontrer, que pour

*) Mathematische Annalen, Band VI, S. 366.



Aigner (2013)

Reutenauer (2018)

Markoff numbers

Def A Markoff triplet is a positive integer solution to the equation

$$x^2+y^2+z^2=3xyz$$

EX: $(1,1,1)$, $(1,1,2)$, $(1,2,5)$, ...

Def An integer is a Markoff number if it appears in a Markoff triplet.

EX: $1, 2, 5, \ldots$ are Markoff numbers

Lemma If (x,y,z) is a M. triplet, then $(3yz-x,y,z)$ also.

Proof

$$(3yz-x)^2+y^2+z^2 = 9y^2z^2-6xyz+x^2+y^2+z^2$$
$$= 9y^2z^2-3xyz = 3(3yz-x)yz.$$

Also, $3yz-x = \dfrac{y^2+z^2}{x} > 0$. $\square$

Assuming $x<y<z$, then we have 3 other triplets:

$$\begin{array}{ccc}
 & (3xy-z,\, x,\, \underline{y}) \text{ with } 3xy-z<y & \\
 & \uparrow & \\
 & (x<y<\underline{z}) & \\
\swarrow & & \searrow \\
(x<z<\underline{3xz-y}) & & (y<z<\underline{3yz-x})
\end{array}$$

Ref: Markoff, 1880 p. 397
or Reutenauer Lemma 3.1.6

the maximum of a triplet is underlined

Because $3xz-y>3xz-z=(3x-1)z>z$

Because $3yz-x>3yz-z=(3y-1)z>z$.

The tree of Markoff triplets

Proposition (Markoff, 1880) Every Markoff triplet appears in the tree

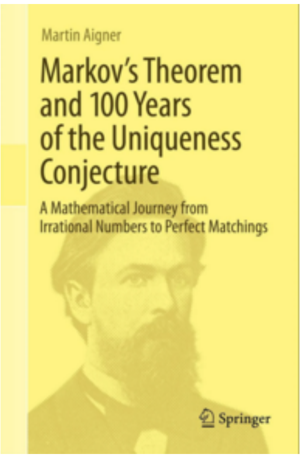
(1, 1, 1) — (1, 1, 2)

(1, 2, 5)

(1, 5, 13) (2, 5, 29)

(1, 13, 34) (5, 13, 194) (5, 29, 433) (2, 29, 169)

... ...



The Uniqueness Conjecture (Frobenius, 1913)

Every Markoff number is the maximum of a unique Markoff triplet.

A motivation for the Conjecture (Dio. Approximations)

Theorem Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then $\exists \infty$ $p/q \in \mathbb{Q}$ distinct s.t.

$$\left|\alpha - \frac{p}{q}\right| < \begin{cases} \frac{1}{q^2} & \text{(Dirichlet, 1842)} \\ \frac{1}{\sqrt{5}\,q^2} & \text{(Horwitz, 1891)} \\ \frac{1}{\sqrt{8}\,q^2} & \text{if } \alpha = [a_0; a_1, a_2, a_3 \dots] \text{ with } \#\{n : a_n \geq 2\} = \infty \\ \dots & \dots \end{cases}$$

Lagrange Spectrum

$$L(\alpha) = \sup\left\{L : \left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2} \text{ for } \infty\text{-many } p/q \in \mathbb{Q}\right\}$$

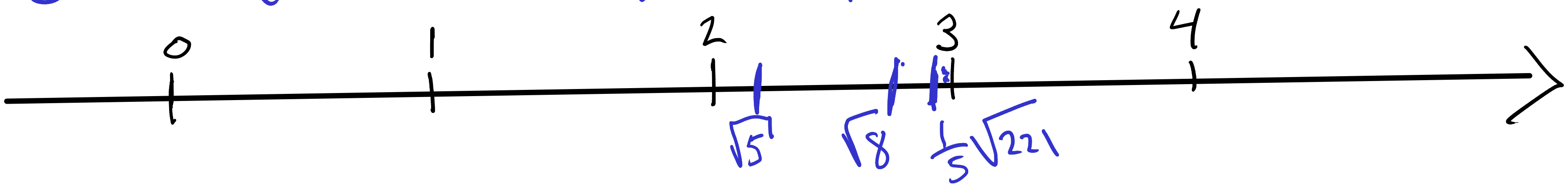
$$\mathcal{L} = \{L(\alpha) : \alpha \in \mathbb{R}\setminus\mathbb{Q}\}$$

Rmk: $\alpha \in \mathbb{Q} \iff L(\alpha) = 0$

$\alpha \notin \mathbb{Q} \iff L(\alpha) \geq \sqrt{5}$

Theorem (Markoff, 1879, 1880)

$$\mathcal{L} \cap (-\infty, 3) = \left\{\sqrt{9 - \frac{4}{m^2}} : m \text{ is a Markoff number}\right\}$$



$$L\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5} = L\left(\frac{3+\sqrt{5}}{2}\right) = \dots$$

$$L\left(\sqrt{2}+1\right) = \sqrt{8} = L\left(2\sqrt{2}+3\right) = \dots$$

<u>Definition</u> $\alpha, \beta \in \mathbb{R}\setminus\mathbb{Q}$ We write $\alpha \sim \beta$ if their continued fraction expansion eventually coincide

i.e. $\alpha = [a_0; a_1, a_2, \dots, a_k, \gamma]$ and

$\beta = [b_0; b_1, b_2, \dots, b_\ell, \gamma]$ for some $\gamma \in (\mathbb{N}_{>0})^{\mathbb{N}}$.

<u>Conjecture</u> Let $\alpha, \beta \in \mathbb{R}\setminus\mathbb{Q}$ with $L(\alpha), L(\beta) < 3$

$$\alpha \sim \beta \iff L(\alpha) = L(\beta).$$

<u>Note</u>: This conjecture is equivalent to the uniqueness conjecture

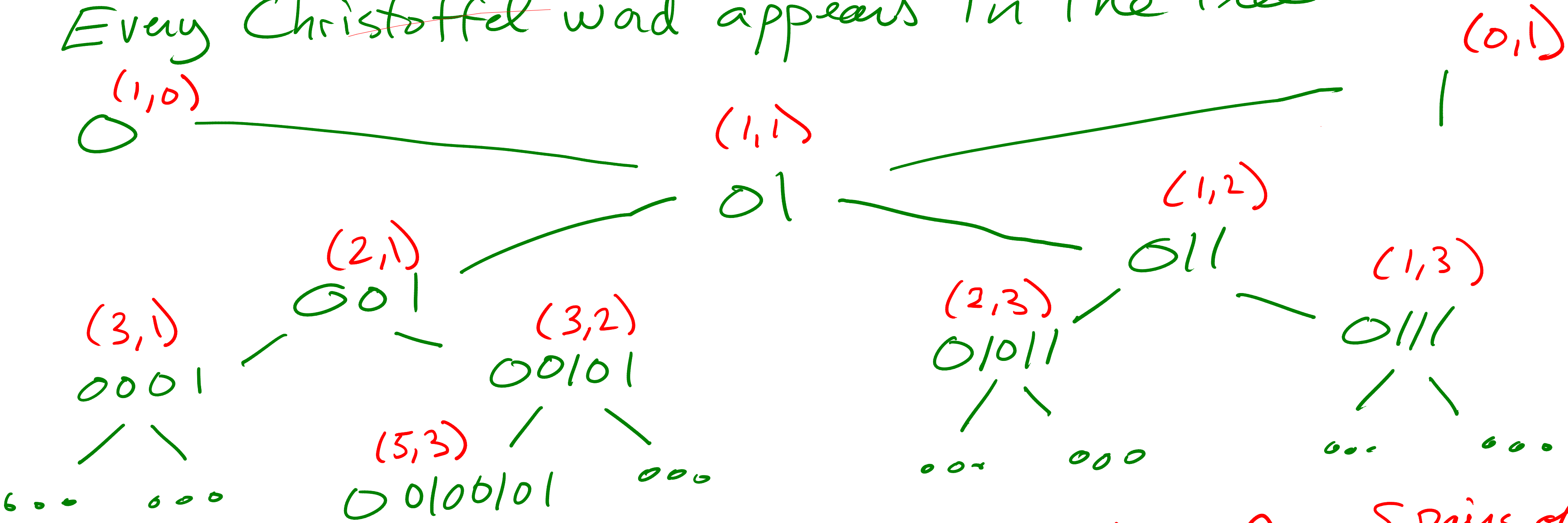
Def The set of Christoffel words is the smallest set $C \subset \{0,1\}^*$ s.t. $0 \in C$, $1 \in C$, $01 \in C$ and if $u, v, uv \in C$, then $uuv, uvv \in C$.

Ex:

- $0, 1, 01 \in C \Rightarrow 001, 011 \in C$
- $0, 01, 001 \in C \Rightarrow 0001, 00101 \in C$
- $01, 1, 011 \in C \Rightarrow 01011, 0111 \in C$

The tree of Christoffel words

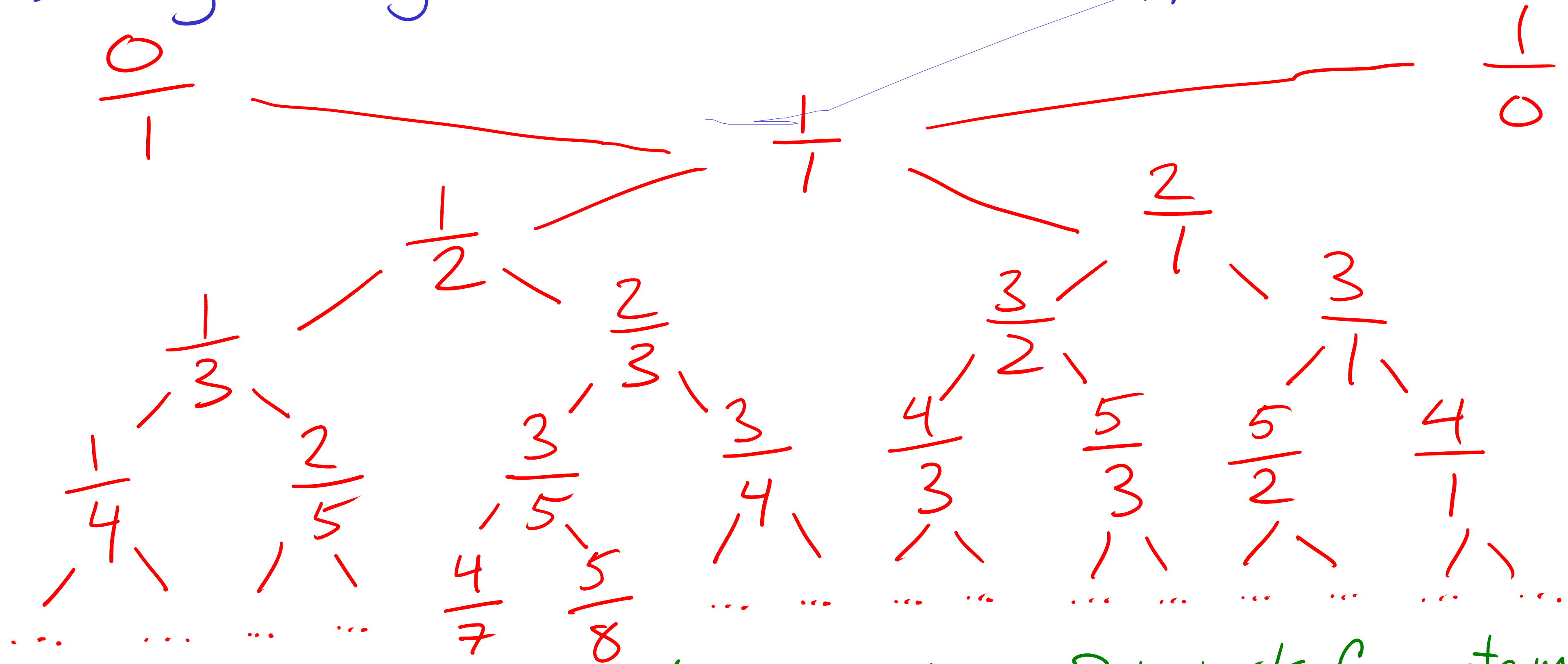
Every Christoffel word appears in the tree



The map $w \mapsto (|w|_0, |w|_1)$ is a bijection $C \to \left\{ \begin{array}{l} \text{pairs of} \\ \text{coprime} \\ \text{integers} \end{array} \right\}$

The Stern Brocot tree

Every nonnegative rational number appears once in the tree



and exactly once (Graham, Knuth, Patashnik, Concrete mathematics).

(Berstel, Lauve, Reutenauer, Saliola, 2009) The map $w \longmapsto \frac{|w|_1}{|w|_0}$ is an isomorphism from the tree of Christoffel words to the Stern-Brocot tree.

Markoff injectivity conjecture

Let $\mu : \{0,1\}^* \to SL_2(\mathbb{Z})$ be the homomorphism defined by $\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$.

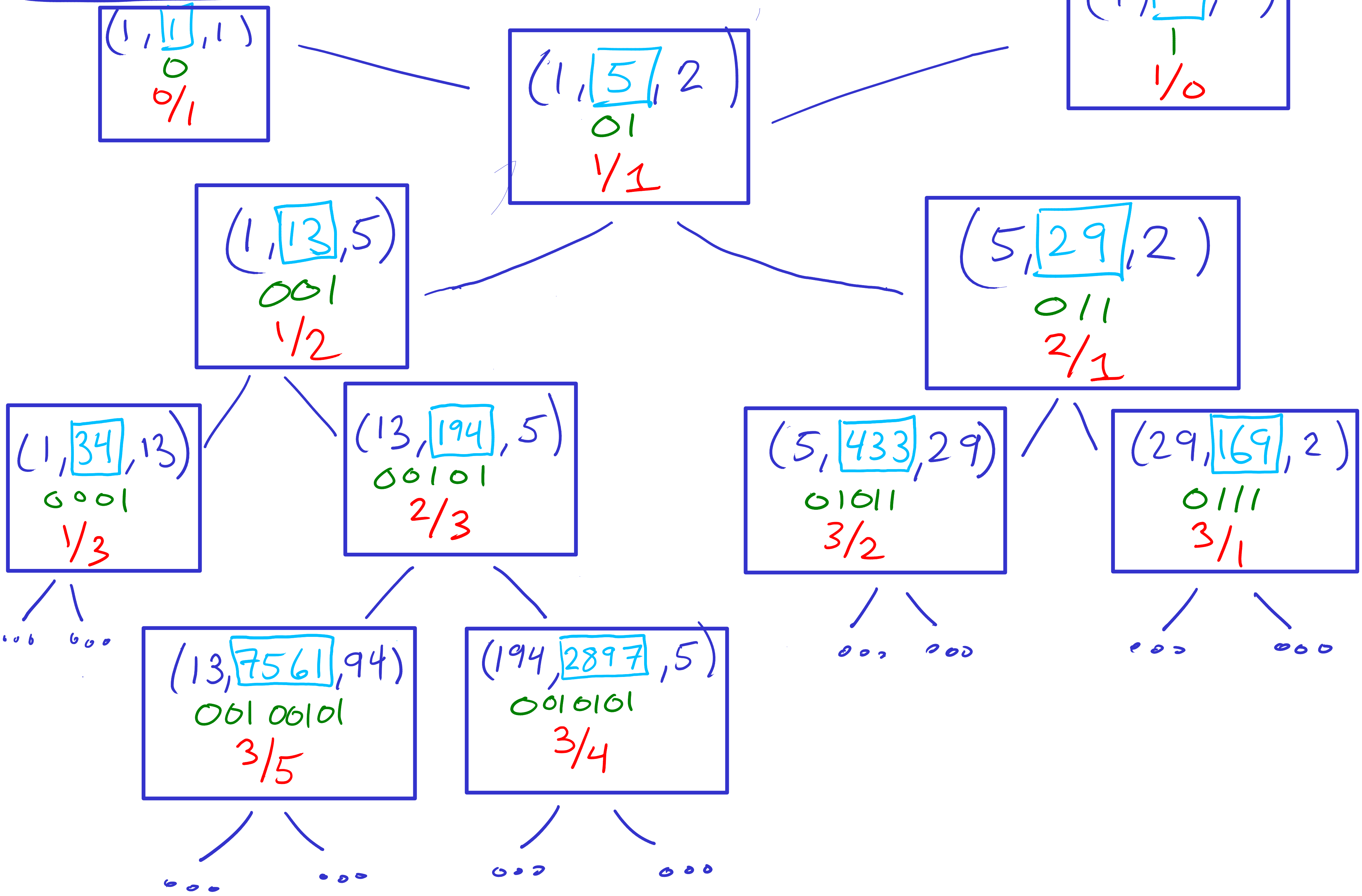
Reutenauer (2009) For every Markoff number m, there exists a Christoffel word $w \in C$ such that $m = \mu(w)_{12} := (1\ 0)\,\mu(w)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Ex: $\mu(00101) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix}$

Markoff Injectivity Conjecture ($\equiv$ the uniqueness conjecture)

The map $w \mapsto \mu(w)_{12}$ is injective over the set C of Christoffel words.

The three Trees Together



q-analogs

A q-analog of Something is Something_q such that

$$\lim_{q\to 1} \text{Something}_q = \text{Something}.$$

EX The q-analog of an integer $n \in \mathbb{N}$ is

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\dots+q^{n-1} \quad \text{and} \quad \lim_{q\to 1} [n]_q = n$$

EX The q-analog of the factorial $n!$ is

$$[n]_q! = [1]_q [2]_q \cdots [n]_q \quad \text{and} \quad \lim_{q\to 1} [n]_q! = n!$$

We have

$$n! = \# S_n = \#\{\text{permutations of } n \text{ elements}\} = \sum_{\sigma \in S_n} 1$$

$$[n]_q! = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)}$$

where $\mathrm{inv}(\sigma) = \#\text{inversions in the permutation } \sigma$

q-analogues of rational numbers

For every $\frac{r}{s} \in \mathbb{Q}_{>0}$, $\exists!$ integers $a_0 \geq 0$, $a_i > 0$, n odd s.t.

$$\frac{r}{s} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = [a_0; a_1, a_2, \ldots, a_n] \quad (n \text{ odd})$$

Also,

$$\begin{pmatrix} r \\ s \end{pmatrix} = R^{a_0} L^{a_1} \cdots R^{a_{n-1}} L^{a_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where } R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(Morier-Genoud, Ovsienko 2020) Proposed:

$$\begin{pmatrix} R(q) \\ S(q) \end{pmatrix} = q^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{n-1}} L_q^{a_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where } R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \; L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$$

and

$$\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}.$$

<u>EX</u>

$$\left[\frac{2}{7}\right]_q = \frac{q^4 + q^3}{q^4 + 2q^3 + 2q^2 + q + 1} \qquad \lim_{q \to 1} \left[\frac{2}{7}\right]_q = \frac{2}{7}.$$

A q-analogue of Markoff numbers (Kogiso, 2020)

We have $\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = RL$

and $\mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = RRLL$ with $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Let $L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$ and $R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$ where q is some indeterminate.

Let $\mu_q(0) = R_q L_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}$

$\mu_q(1) = R_q R_q L_q L_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$

which extends to a homomorphism $\mu_q : \{0,1\}^* \to GL_2(\mathbb{Z}[q, q^{-1}])$

<u>Def</u>: If $w \in C$ is a Christoffel word, then $\mu_q(w)_{12}$ a <u>q-analogue of a Markoff number</u>.

<u>EX</u>: $\mu_q(00101)_{12} = 1+4q+10q^2+18q^3+27q^4+33q^5+33q^6+29q^7+21q^8+12q^9+5q^{10}+q^{11}$

which evaluates at 194 when $q=1$.

Let $(\{0,1\}^*, <_{radix})$ be a total order s.t.

$$u <_{radix} v \quad \text{if} \begin{cases} |u| < |v| \\ |u| = |v| \text{ and } \quad u <_{lex} v \end{cases}$$

Let $(\mathbb{Z}[q], \prec)$ be a partial order on polynomials s.t.

$$f \prec g \iff f \neq g \text{ and } g - f \in \mathbb{Z}_{\geq 0}[q]$$

<u>Theorem</u> (L, Lapointe, 2022)

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence.

For every $u, v \in \mathcal{L}(s)$

$$u <_{radix} v \implies \mu_q(u)_{12} \prec \mu_q(v)_{12}.$$

<u>Corollary</u> $w \mapsto \mu_q(w)_{12}$ is injective over the language $\mathcal{L}(s)$ of a balance sequence $s \in \{0,1\}^{\mathbb{Z}}$.

<u>Corollary</u> (Lapointe, Reutenauer, 2021) $w \mapsto \mu(w)_{12}$ is injective the language of a balanced sequence.

Theorem (L, Lapointe, Steiner, 2023)

The map $w \mapsto \mu_q(w)_{12}$ is injective over the set C of Christoffel words.

Note: this is of course weaker than the Markoff injectivity conjecture because distinct polynomials can evaluate at the same value at $q=1$.

Idea of the Proof: Evaluating the polynomials $\mu_q(w)_{12}$ at the sixth root of unity allows to recover the number of occurrences of 0 and 1 in w, thus the rationnal $\frac{|w|_1}{|w|_0}$.

Injectivity follows from the isomorphism between the Christoffel tree and the Stern-Brocot tree, because no rational number appears twice in the Stern-Brocot tree.

Idea of the proof (not mentionned in the published version)

Let $z = e^{i\frac{\pi}{3}}$ root of $x^2 - x + 1$. then

$$A = \mu_q(0)_{1,2}\Big|_{q=z} = \begin{pmatrix} 2z^{-1} & 1 \\ z & 1 \end{pmatrix}.$$

$$B = \mu_q(1)_{1,2}\Big|_{q=z} = \begin{pmatrix} 2z-3 & z+1 \\ 2z-1 & 1 \end{pmatrix}$$

and

$$AB = \begin{pmatrix} -2z-2 & 3z-2 \\ z-3 & 2z \end{pmatrix} = BA.$$

Thus

$$\mu_q(w) = A^{|w|_0} B^{|w|_1}.$$

Also $\langle A, B \rangle \cong \mathbb{Z}^2$

Remark The map $w \mapsto \mu_q(w)_{12}$ is not injective over $\{0,1\}^*$.

Ex $\mu_q(00011)_{12} = 1 + 4q + 10q^2 + 19q^3 + 27q^4 + 33q^5 + 34q^6$
$+ 29q^7 + 21q^8 + 12q^9 + 5q^{10} + q^{11}$
$= \mu_q(01001)_{12}$

In general, we have

Lemma (L, Lapointe, Steiner) $\forall w \in \{0,1\}^*$ $\mu_q(0w1)_{12} = \mu_q(0\tilde{w}1)_{12}$

where $\tilde{w} = w_n \cdots w_2 w_1$ denotes the reversal of $w = w_1 w_2 \cdots w_n$.

Amnisty International Report, December 5th, 2024

#stopgazagenocide

'YOU FEEL LIKE YOU ARE SUBHUMAN'

ISRAEL'S GENOCIDE AGAINST PALESTINIANS IN GAZA

https://www.amnesty.org/en/documents/mde15/8668/2024/en/