Metallic mean Wang tiles and their Rauzy fractals

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Uniform distribution of Sequences
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`https://www.esi.ac.at/events/e577/`

Outline

1. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles

2. Metallic mean Wang tiles

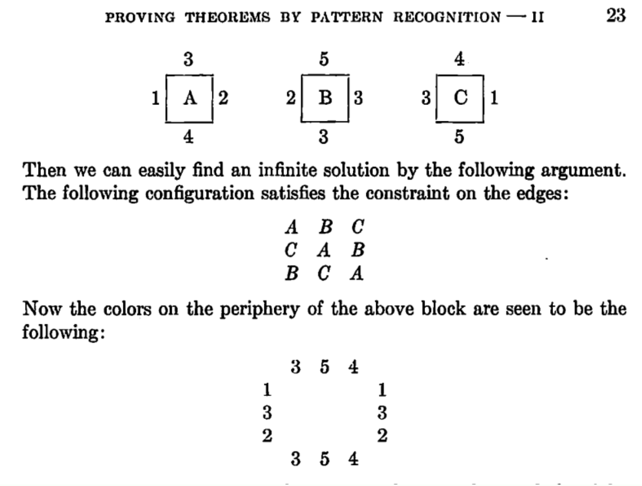
3. Stone inflations and 4-to-2 Rauzy fractals

4. Open questions

Outline

1. **Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles**
2. Metallic mean Wang tiles
3. Stone inflations and 4-to-2 Rauzy fractals
4. Open questions

Wang tiles



PROVING THEOREMS BY PATTERN RECOGNITION — II 23

Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

A B C
C A B
B C A

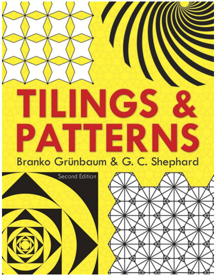
Now the colors on the periphery of the above block are seen to be the following:

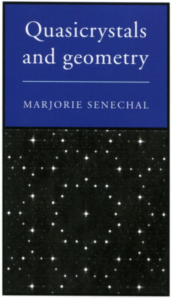
3 5 4
1 1
3 3
2 2
3 5 4

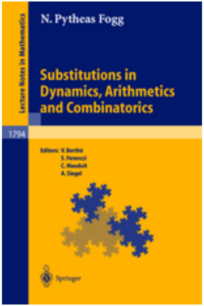
Wang's original question : is it true that a set of Wang tiles tile the plane if and only if there exists such a cyclic rectangle ?

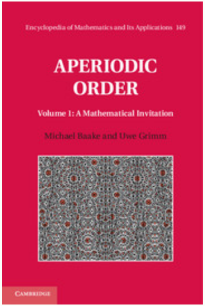
H. Wang. Proving theorems by pattern recognition – II. Bell System Technical Journal*, 40(1) :1–41, January 1961.* `doi:10.1002/j.1538-7305.1961.tb03975.x`

Books



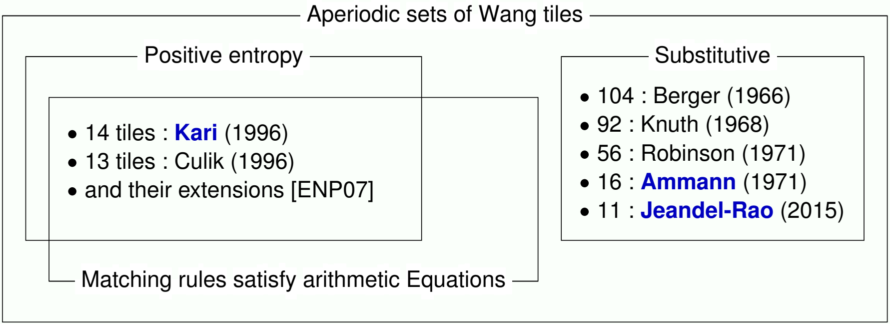






- Tilings and Patterns, by Grünbaum & Shephard, 1987
- Quasicrystals and Geometry, Senechal, 1995
- Pytheas Fogg's book, 2002
- Aperiodic Order, Baake & Grimm, 2013

Aperiodic Wang tile sets



Aperiodic sets of Wang tiles

Positive entropy

Matching rules satisfy arithmetic Equations

- 14 tiles : **Kari** (1996)
- 13 tiles : Culik (1996)
- and their extensions [ENP07]

Substitutive

- 104 : Berger (1966)
- 92 : Knuth (1968)
- 56 : Robinson (1971)
- 16 : **Ammann** (1971)
- 11 : **Jeandel-Rao** (2015)

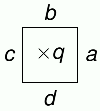
Theorem (Jeandel, Rao, 2015)

All sets of $\leq$ 10 Wang tiles are **periodic** or **don’t tile** the plane.

Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. Adv. Comb. ***37*** *(2021) Id/No 1.*

Kari’s 14 Wang tiles computing $\times\frac{2}{3}$ and $\times 2$

2 -1/3 0/3 1	2 0/3 1/3 1	2 1/3 2/3 1	2 1/3 -1/3 2	2 2/3 0/3 2	1 0/3 -1/3 1	1 1/3 0/3 1	1 2/3 1/3 1	1 -1/3 1/3 0	1 0/3 2/3 0
1 -1 -1 2	1 -1 0 1	0 0 -1 1	1 0 0 2						

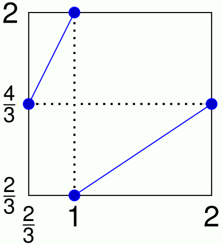


$$\iff qb + c = d + a$$

with $q \in \{\frac{2}{3}, 3\}$

$$g(x) = \begin{cases} 2x & \text{if } x \le 1, \\ \frac{2}{3}x & \text{if } x > 1. \end{cases}$$

Averages of horizontal labels are orbits of g :

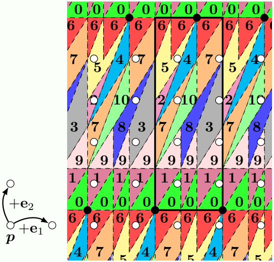


1 1/3 6 0/3 1	1 0/3 5 -1/3 1	2 -1/3 0 0/3 1	1 0/3 9 2/3 0	1 2/3 7 1/3 1	1 1/3 6 0/3 1	1 0/3 9 2/3 0	1 2/3 7 1/3 1
1 -1 11 0 1	1 0 13 0 2	1 0 13 0 2	0 0 12 -1 1	1 -1 11 0 1	1 0 13 0 2	0 0 12 -1 1	1 -1 10 -1 2
1 0/3 5 -1/3 1	2 -1/3 0 0/3 1	2 0/3 1 1/3 1	1 1/3 6 0/3 1	1 0/3 5 -1/3 1	2 -1/3 0 0/3 1	1 0/3 5 -1/3 1	2 -1/3 0 0/3 1
1 -1 10 -1 2	1 -1 10 -1 2	1 -1 10 -1 2	1 -1 10 -1 2	1 -1 10 -1 2	1 -1 10 -1 2	1 -1 10 -1 2	1 -1 10 -1 2

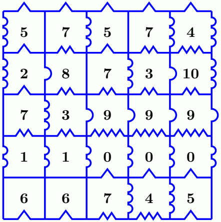
$$\begin{tikzcd}
\frac{9}{8} \arrow[d, "\times\frac{2}{3}"] \\
\frac{3}{4} \arrow[d, "\times 2"] \\
\frac{3}{2} \arrow[d, "\times\frac{2}{3}"] \\
1 \arrow[d, "\times 2"] \\
2
\end{tikzcd}$$

Durand, Gamard, Grandjean (2007) *Kari (2016)*

Jeandel–Rao aperiodic set of 11 Wang tiles



$$\xrightarrow[\text{repr.}]{\text{symb.}} \begin{pmatrix} 5 & 7 & 5 & 7 & 4 \\ 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \end{pmatrix} \xrightarrow[\text{tiling}]{\text{Wang}}$$



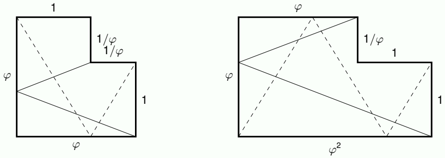
Markov partitions for toral $\mathbb{Z}^2$-rotations featuring Jeandel-Rao Wang shift and model sets. Ann. H. Lebesgue *4 (2021) 283–324.* `doi:10.5802/ahl.73`

Rauzy induction of polygon partitions and toral $\mathbb{Z}^2$-rotations. J. Mod. Dyn. *17 (2021) 481–528.* `doi:10.3934/jmd.2021017`

Substitutive structure of Jeandel-Rao aperiodic tilings. Discrete Comput. Geom., *65 (2021) 800–855.* `doi:10.1007/s00454-019-00153-3`

Open question. Jeandel–Rao is not alone : characterize the family.

Ammann A2 encoded into 16 Wang tiles



Tilings in the Ammann A2 family can be encoded into 16 Wang tiles :

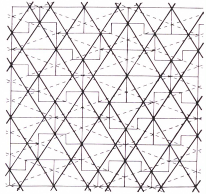


Figure 11.1.10
A tiling by the set A2 of Ammann prototiles with the four families of Ammann bars indicated, two by solid and two by dashed lines.

Figure 11.1.10

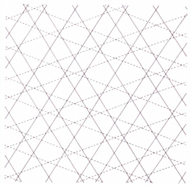


Figure 11.1.11
The Ammann bars of Figure 11.1.10 after the tiles have been detected. The solid bars are to be regarded as the edges of a new tiling by rhombs and parallelograms, the dashed bars are to be regarded as markings on the tiles specifying the matching condition.

Figure 11.1.11

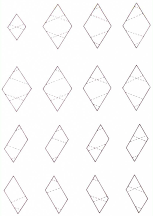


Figure 11.1.12
The 16 tiles that arise as indicated in Figure 11.1.11.

Figure 11.1.12

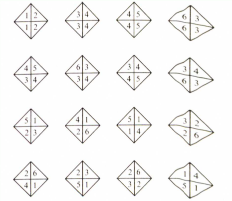
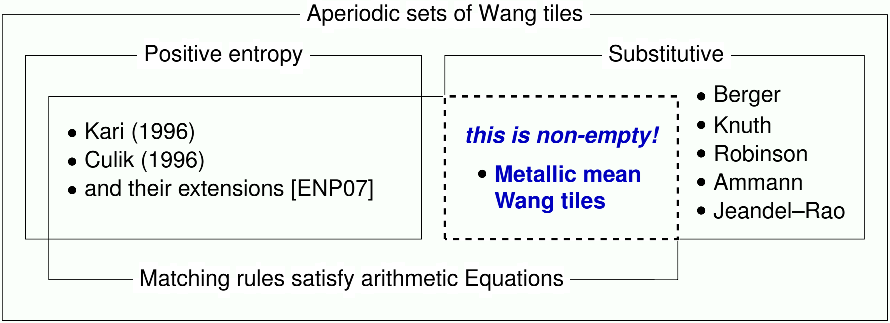


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12. These form the smallest known aperiodic set.

Figure 11.1.13

Branko Grünbaum and G. C. Shephard. Tilings and patterns. *W. H. Freeman and Company, New York, 1987.*

This talk's take home message



Aperiodic sets of Wang tiles

Positive entropy

Substitutive

Matching rules satisfy arithmetic Equations

- Kari (1996)
- Culik (1996)
- and their extensions [ENP07]

this is non-empty!

- **Metallic mean Wang tiles**

- Berger
- Knuth
- Robinson
- Ammann
- Jeandel–Rao

... and metallic mean Wang tiles have **lots of nice properties !**

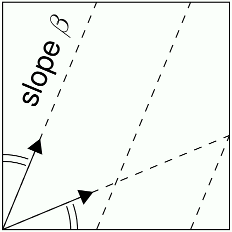
Outline

1. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles
2. **Metallic mean Wang tiles**
3. Stone inflations and 4-to-2 Rauzy fractals
4. Open questions

Metallic means

Definition

The ***n*-th metallic mean** is the positive root of $x^2 - nx - 1$.



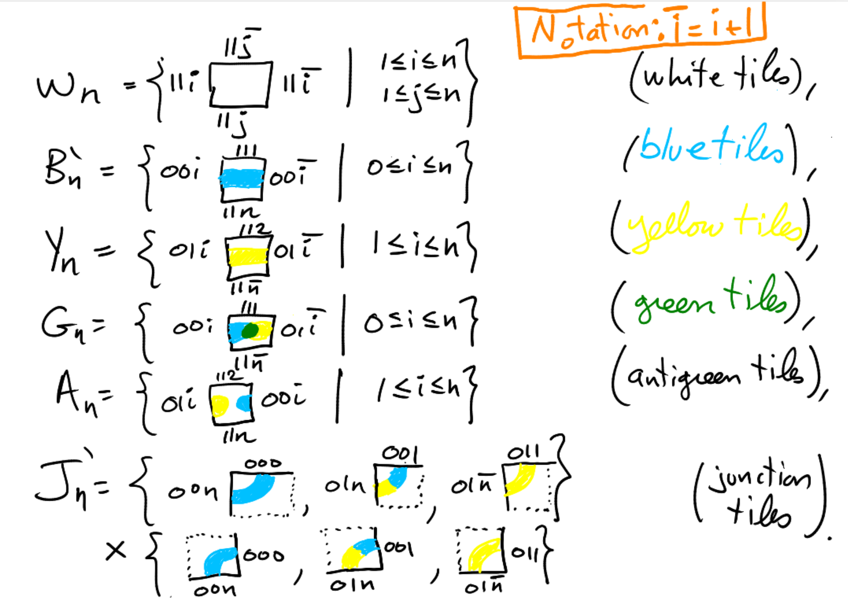
$$\beta = \frac{n + \sqrt{n^2 + 4}}{2} = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\cdots}}}$$

`https://oeis.org/wiki/Metallic_means`

V. W. de Spinadel. The family of metallic means. Vis. Math., *1(3) :1 HTML document ; approx. 16, 1999.*

Also called ***silver means*** *(Schroeder 1991) or* ***noble means*** *(Baake, Grimm, 2013).*

Metallic mean Wang tiles



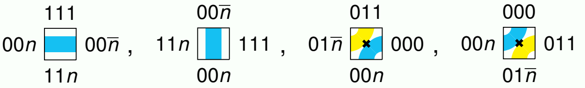
A family $\{\mathcal{T}_n\}_{n\geq 1}$ of metallic mean Wang tiles

An **extended set** $\mathcal{T}'_n$ of metallic mean Wang tiles :

$$\mathcal{T}'_n = W_n \cup B'_n \cup G_n \cup Y_n \cup A_n \cup \widehat{B'_n} \cup \widehat{G_n} \cup \widehat{Y_n} \cup \widehat{A_n} \cup J'_n.$$

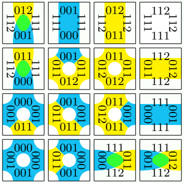
The following tiles are **non-extendible** (proof is not trivial) :

$\mathcal{D} = A_n \cup \widehat{A_n} \cup \Bigg\{$

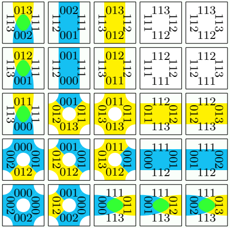


$\Bigg\}$.

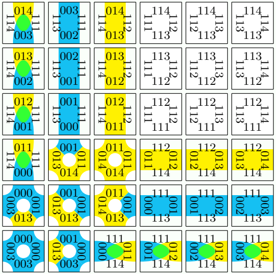
The **subset $\mathcal{T}_n = \mathcal{T}'_n \setminus \mathcal{D}$ of metallic mean Wang tiles** contains $(n+3)^2$ tiles :



$\mathcal{T}_1$



$\mathcal{T}_2$

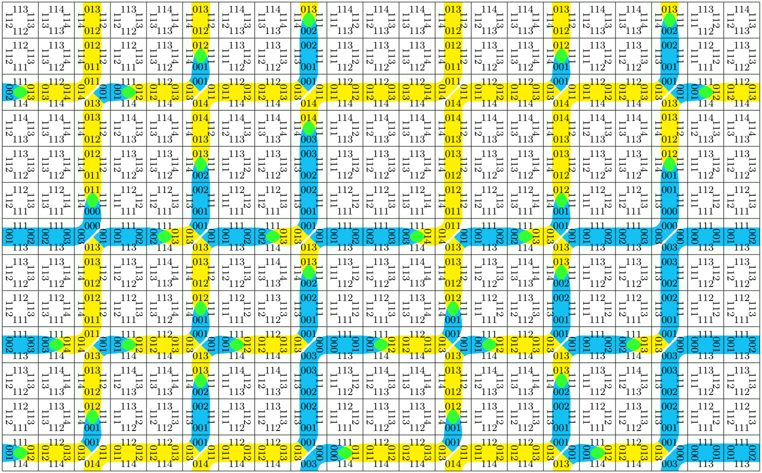


$\mathcal{T}_3$

Metallic mean Wang shift

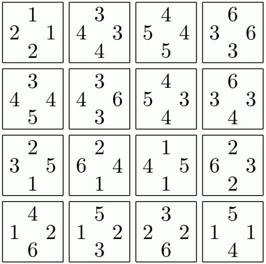
The ***n*-th metallic mean Wang shift** is $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ where

$$\Omega_n := \Omega_{\mathcal{T}_n} = \{w : \mathbb{Z}^2 \to \mathcal{T}_n \colon w \text{ is a valid configuration}\}.$$

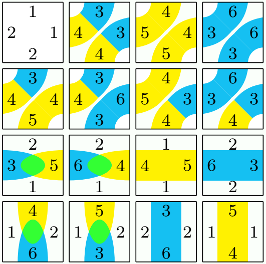


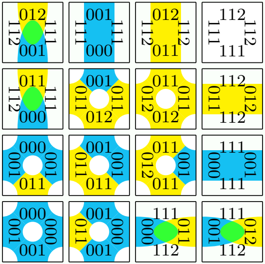
(a 21 × 13 valid patch with $\mathcal{T}_3$)

Ammann $\equiv \mathcal{T}_1$



Ammann





$\mathcal{T}_1$

Theorem

The Ammann set of 16 Wang tiles is equivalent to $\mathcal{T}_1$.

Proof : the bijection of the tile labels is

$$1 \mapsto 112, \quad 2 \mapsto 111, \quad 3 \mapsto 001, \quad 4 \mapsto 011, \quad 5 \mapsto 012, \quad 6 \mapsto 000.$$

Self-similarity, aperiodicity and minimality

Theorem

For every integer $n \geq 1$, the metallic mean Wang shift Ω_n is

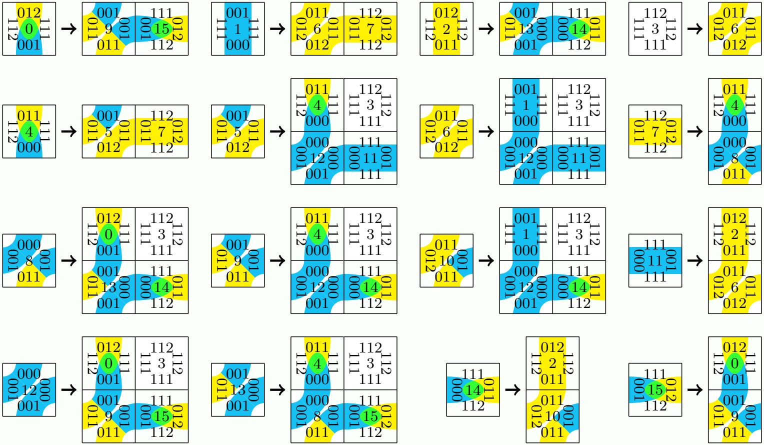
- **self-similar**,
- **aperiodic** and
- **minimal**.

The inflation factor of the self-similarity of Ω_n is the n-th metallic mean, that is, the positive root of $x^2 - nx - 1$.

Self-similarity proof (main idea) : the set of **return blocks** to the junction tiles J_n is in bijection with the **extended set** $\mathcal{T}'_n$.

Metallic mean Wang tiles I : self-similarity, aperiodicity and minimality.
`arXiv:2312.03652`

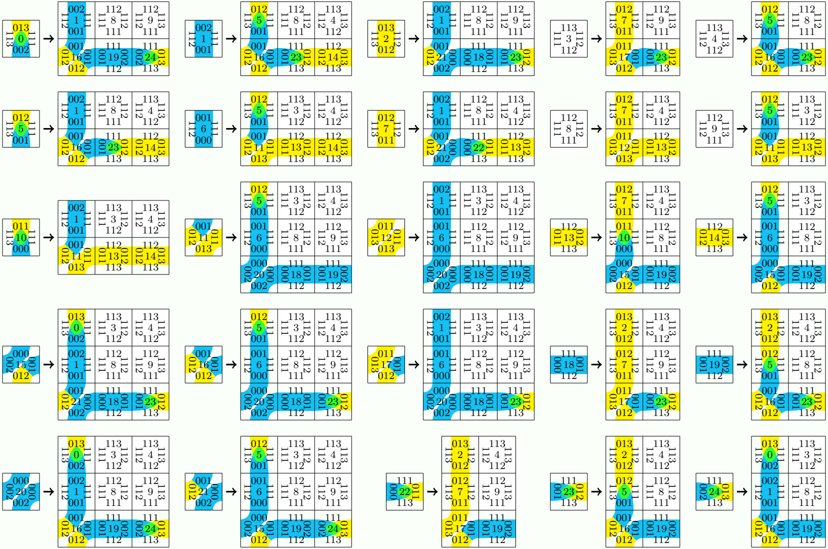
Substitution $\omega_1 : \Omega_1 \to \Omega_1$



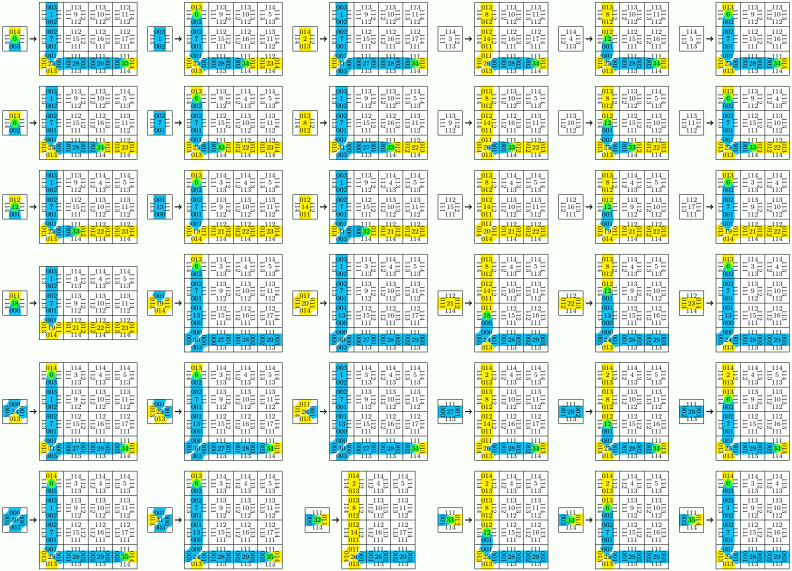
The self-similarity is a non-uniform rectangular 2-dimensional substitution as in Mozes (1989).

Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. J. Analyse Math., *53 :139–186, 1989.*

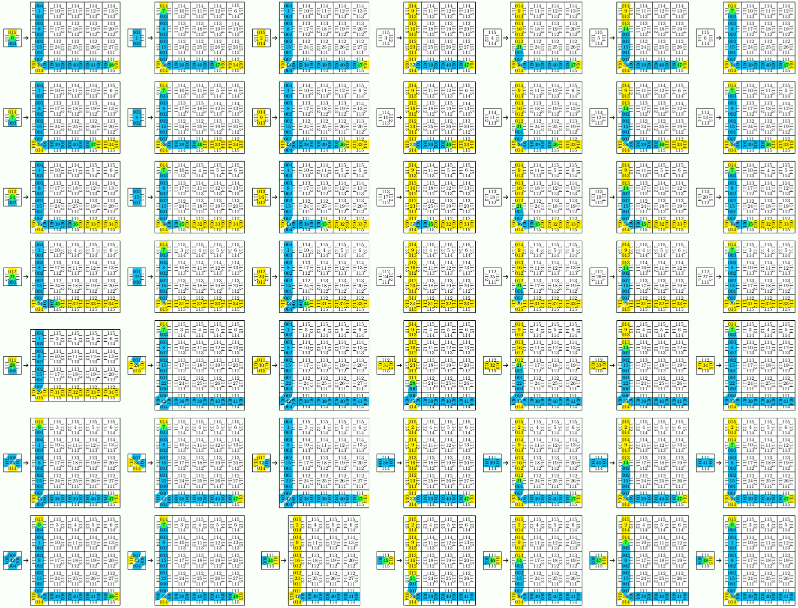
Substitution $\omega_2 : \Omega_2 \to \Omega_2$



Substitution $\omega_3 : \Omega_3 \to \Omega_3$



Substitution $\omega_4 : \Omega_4 \to \Omega_4$



The θ_n-computer chip

For every integer $n \geq 1$, we define the finite set of vectors

$$V_n = \{(v_0, v_1, v_2) \in \mathbb{N}^3 \colon 0 \leq v_0 \leq v_1 \leq 1 \text{ and } v_1 \leq v_2 \leq n+1\}$$

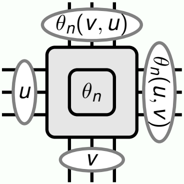
with **nondecreasing** entries. Let

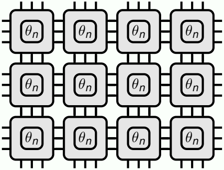
$$\begin{array}{rccc} \boldsymbol{\theta_{\mathbf{n}}} : & V_n \times V_n & \to & \mathbb{Z}^3 \\ & (u_0, u_1, u_2), (v_0, v_1, v_2) & \mapsto & (r_0, r_1, r_2), \end{array}$$

be the **map** defined by the rule

$$r_0 = u_0, \quad r_1 = \begin{cases} v_2 - n & \text{if } u_0 = 0, \\ 1 & \text{if } u_0 = 1, \end{cases}, \quad r_2 = \begin{cases} v_1 + u_0 & \text{if } v_0 = 0, \\ u_2 + 1 & \text{if } v_0 = 1. \end{cases}$$

The **θ_n-chip is a computer chip** computing $\theta_n(u, v)$ and $\theta_n(v, u)$ from the left input u and bottom input v :





The θ_n-computer chip

Let

$$\mathcal{C}_n = \left\{ u\ \underset{v}{\overset{\theta_n(v,u)}{\boxed{\boxed{\theta_n}}}}\ \theta_n(u,v) \;\middle|\; \begin{array}{l} u,v \in V_n \text{ and} \\ \theta_n(u,v), \theta_n(v,u) \in V_n \end{array} \right\}$$

be the finite set of all possible instances of the θ_n-chip **with outputs restricted to V_n**.

Theorem

For every integer $n \geq 1$, the Wang shift $\Omega_{\mathcal{C}_n}$ defined by the θ_n-chip **is the n^{th} metallic mean Wang shift Ω_n**.

Proof : $\mathcal{C}_n = \mathcal{T}'_n$ is the **extended set** of metallic mean Wang tiles (case by case analysis).

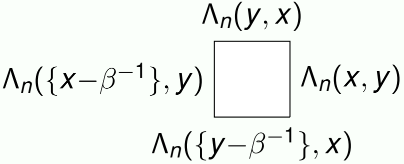
Metallic mean Wang tiles II : the dynamics of an aperiodic computer chip.
`arXiv:2403.03197`

Existence of valid tilings

For every $(x, y) \in [0, 1)^2$, let $\Lambda_n(x, y) = \begin{pmatrix} \lfloor y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta^{-1}x + y - \beta^{-1} + 1 \rfloor \\ \lfloor \beta x + y - \beta^{-1} + 1 \rfloor \end{pmatrix} \in \mathbb{N}^3$

where β is the positive root of the polynomial $x^2 - nx - 1$ and

$t_n(x, y) =$



be a Wang tile

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of a number $x \in \mathbb{R}$.

Theorem

For every integer $n \geq 1$ and every $(x, y) \in [0, 1)^2$,

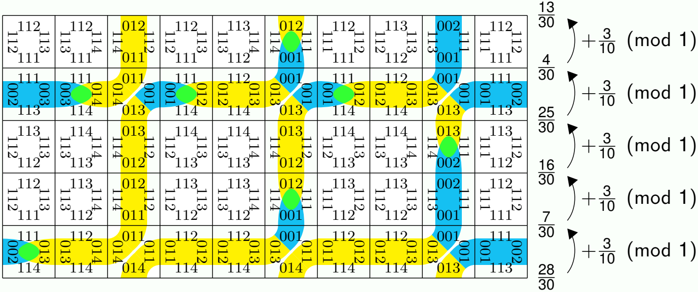
$$\begin{array}{rccl} c_{(x,y)} : & \mathbb{Z}^2 & \to & \mathcal{T}_n \\ & (i, j) & \mapsto & t_n\left(\{x + i\beta^{-1}\}, \{y + j\beta^{-1}\}\right) \end{array}$$

is a **valid tiling** with the **metallic mean** Wang tiles $\mathcal{T}_n$.

An explicit factor map (example)

A 10×5 valid rectangular tiling with the set $\mathcal{T}_n$ with $n = 3$.

The numbers indicated in the right margin are the average of the inner products $\langle \frac{1}{n} d, v \rangle$ over the vectors v appearing as top (or bottom) labels of a horizontal row of tiles and where $d = (0, -1, 1)$.



We observe that these numbers increase by $\frac{3}{10} \pmod 1$ from row to row. The number $\frac{3}{10}$ is equal to the frequency of columns containing junction tiles (a junction tile is a tile whose labels all start with 0).

An explicit factor map

Theorem

Let $d = (0, -1, 1)$, $n \geq 1$ be an integer and Ω_n be the n^{th} metallic mean Wang shift. The map

$$\begin{array}{rccl} \Phi_n : & \Omega_n & \to & \mathbb{T}^2 \\ & w & \mapsto & \displaystyle\lim_{k\to\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \begin{pmatrix} \langle \frac{1}{n} d, \textsc{Right}(w_{0,i}) \rangle \\ \langle \frac{1}{n} d, \textsc{Top}(w_{i,0}) \rangle \end{pmatrix} \end{array}$$

is a factor map commuting the shift $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ with $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ by the equation $\Phi_n \circ \sigma^k = R_n^k \circ \Phi_n$ for every $k \in \mathbb{Z}^2$ where

$$\begin{array}{rlcl} R_n & : \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & \quad (k, x) & \mapsto & R_n^k(x) := x + \beta k \end{array}$$

and $\beta = \frac{n+\sqrt{n^2+4}}{2}$ is the n^{th} metallic mean, that is, the positive root of the polynomial $x^2 - nx - 1$.

Proof uses **Weyl equidistribution thm** (Kuipers, Niederreiter, 1974).
Remark : Φ_n satisfies $\Phi_n(c_{(x,y)}) = (x, y)$.

An isomorphism (mod 0)

Theorem

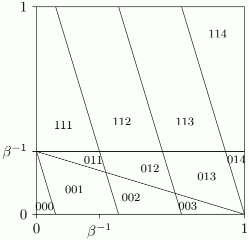
The Wang shift Ω_n and the $\mathbb{Z}^2$-action R_n have the following additional properties :

- $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$ is the **maximal equicontinuous factor** of $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$,
- the factor map $\Phi_n : \Omega_n \to \mathbb{T}^2$ is **almost one-to-one** and its **set of fiber cardinalities** is $\{1, 2, 8\}$,
- the shift-action $\mathbb{Z}^2 \overset{\sigma}{\curvearrowright} \Omega_n$ on the metallic mean Wang shift is **uniquely ergodic**,
- the measure-preserving dynamical system $(\Omega_n, \mathbb{Z}^2, \sigma, \nu)$ is **isomorphic** to $(\mathbb{T}^2, \mathbb{Z}^2, R_n, \lambda)$ where ν is the unique shift-invariant probability measure on Ω_n and λ is the Haar measure on $\mathbb{T}^2$.

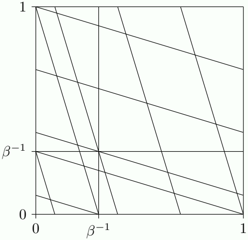
A Markov partition

$\mathcal{P}_n = \{t_n^{-1}(a)\}_{a \in \mathcal{T}_n}$ partitions the unit square into $(n+3)^2$ polygons.

$\{\Lambda_3^{-1}(v)\}_{v \in V_3}$



$\mathcal{P}_3$



Theorem

For every integer $n \geq 1$, the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_n, R_n}$ corresponding to $\mathcal{P}_n, R_n$ **is the metallic mean Wang shift** Ω_n :

$$\Omega_n = \mathcal{X}_{\mathcal{P}_n, R_n}.$$

In particular, $\mathcal{P}_n$ is a **Markov partition** for $\mathbb{Z}^2 \overset{R_n}{\curvearrowright} \mathbb{T}^2$.

Outline

1. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles
2. Metallic mean Wang tiles
3. **Stone inflations and 4-to-2 Rauzy fractals**
4. Open questions

Perron-Frobenius

Let $n \geq 1$ be an integer and $\rho_n : \{\mathsf{a}, \mathsf{b}\}^* \to \{\mathsf{a}, \mathsf{b}\}^*$ be $\begin{cases} \mathsf{a} \mapsto \mathsf{ab}^n \\ \mathsf{b} \mapsto \mathsf{ab}^{n-1} \end{cases}$.

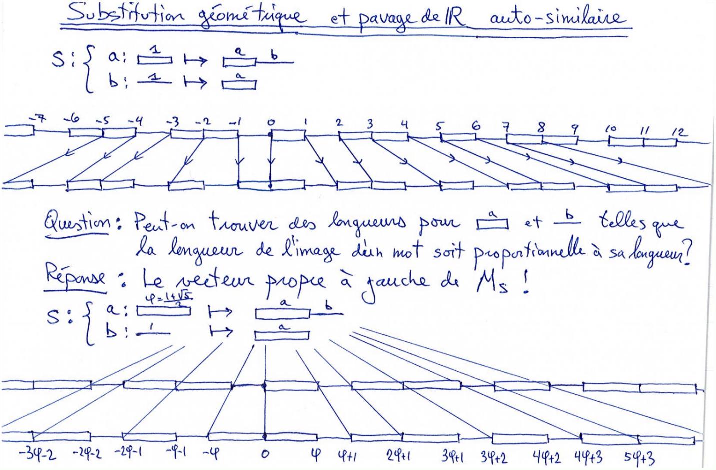
We have

- Incidence matrix of ρ_n is $\left(\begin{smallmatrix} 1 & 1 \\ n & n-1 \end{smallmatrix}\right)$.
- Characteristic polynomial is $x^2 - nx - 1$.
- Perron–Frob. dominant eigenvalue is the n^{th} metallic mean β.
- A right dominant eigenvector : $\left(\begin{smallmatrix} 1 & 1 \\ n & n-1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ \beta-1 \end{smallmatrix}\right) = \beta \left(\begin{smallmatrix} 1 \\ \beta-1 \end{smallmatrix}\right)$.
- A left dominant eigenvector : $\left(\begin{smallmatrix} n & \beta-1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 1 \\ n & n-1 \end{smallmatrix}\right) = \beta \left(\begin{smallmatrix} n & \beta-1 \end{smallmatrix}\right)$.

Note that another substitution with same charateristic polynomial is $\mathsf{a} \mapsto \mathsf{a}^n\mathsf{b}, \mathsf{b} \mapsto \mathsf{a}$ with incidence matrix $\left(\begin{smallmatrix} n & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

A stone inflation

A **stone inflation** associated with a substitution gives a length to each letter proportionally to the entries of the left PF eigenvector.



Substitution géométrique et pavage de ℝ auto-similaire

Question : Peut-on trouver des longueurs pour a et b telles que la longueur de l'image d'un mot soit proportionnelle à sa longueur ?

Réponse : Le vecteur propre à gauche de M_S !

M. Baake and U. Grimm. Aperiodic Order. Vol. 1*, Cambridge University Press, Cambridge, 2013. (see Inflation tilings in Chapter 6)*

A stone inflation for ρ_n

When $n = 3$, $\rho_3 : \begin{cases} \mathrm{a} \mapsto \mathrm{abbb} \\ \mathrm{b} \mapsto \mathrm{abb} \end{cases}$ defines a **point set** $\Lambda = \Lambda_\mathrm{a} \cup \Lambda_\mathrm{b} \subset \mathbb{Z}[\beta]$ satisfying **length of tile a is n** and **length of tile b is $\beta - 1$** and

$$\begin{cases} \Lambda_\mathrm{a} = \beta\Lambda_\mathrm{a} \cup \beta\Lambda_\mathrm{b} = \beta\Lambda \\ \Lambda_\mathrm{b} = (\beta\Lambda_\mathrm{a} + n) \cup (\beta\Lambda_\mathrm{a} + n + \beta - 1) \cup (\beta\Lambda_\mathrm{a} + n + 2(\beta - 1)) \\ \qquad\quad \cup (\beta\Lambda_\mathrm{b} + n) \cup (\beta\Lambda_\mathrm{b} + n + \beta - 1) \end{cases}$$

The top. closure of the Galois conjugate of the point sets ($W_\mathrm{a} = \overline{\Lambda_\mathrm{a}^\star}$ and $W_\mathrm{b} = \overline{\Lambda_\mathrm{b}^\star}$) satisfy the equations of a **contracting GIFS** :

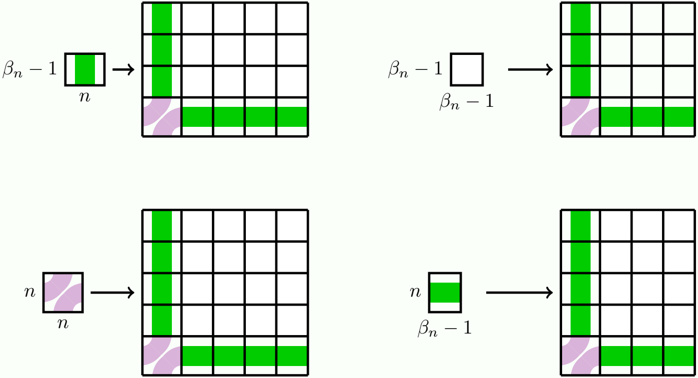
$$\begin{cases} W_\mathrm{a} = \beta^\star W_\mathrm{a} \cup \beta^\star W_\mathrm{b} = \beta^\star W \\ W_\mathrm{b} = (\beta^\star W_\mathrm{a} + n) \cup (\beta^\star W_\mathrm{a} + n + \beta^\star - 1) \cup (\beta^\star W_\mathrm{a} + n + 2(\beta^\star - 1)) \\ \qquad\quad \cup (\beta^\star W_\mathrm{b} + n) \cup (\beta^\star W_\mathrm{b} + n + \beta^\star - 1) \end{cases}$$

whose **unique solution** (Hutchison's theorem) is

$$W_\mathrm{a} = [-1, \beta^{-1}] \qquad \text{and} \qquad W_\mathrm{b} = [\beta^{-1}, \beta].$$

A stone inflation for $\rho_n \times \rho_n$

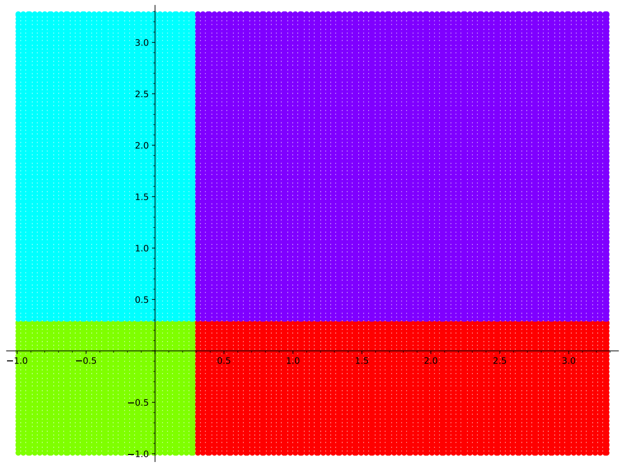
The stone inflation for the direct product $\rho_n \times \rho_n$ over alphabet $\{a \times a, a \times b, b \times a, b \times b\}$ with inflation factor equal to β_n, the n^{th} metallic mean, is



The figure is drawn with parameter $n = 4$. Color is added to the tiles to differentiate them and visually link them to the tiles in $\mathcal{T}_n$.

Rauzy fractal of $\rho_n \times \rho_n$

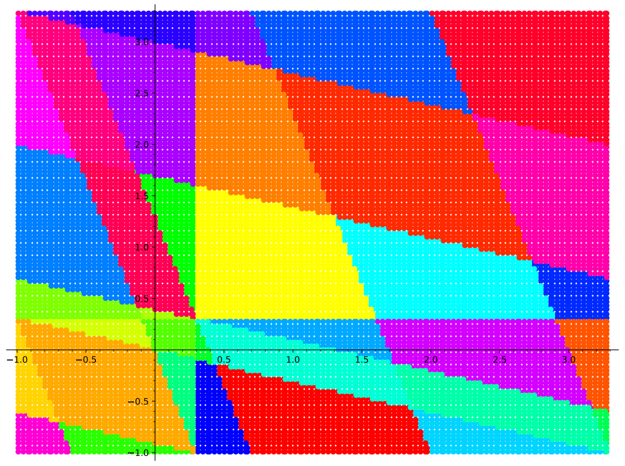
The attractor of the graph iterated function system defined by the Galois conjugate of the stone inflation of $\rho_n \times \rho_n$ is



that is, $\{[-1, \beta^{-1}], [\beta^{-1}, \beta]\} \times \{[-1, \beta^{-1}], [\beta^{-1}, \beta]\}$.

Rauzy fractal of ω_3

The attractor of the graph iterated function system defined by the Galois conjugate of the stone inflation of the self-similarity ω_3 of Ω_3 :



This Rauzy fractal is the image of the earlier Markov partition under the affine transformation $x \mapsto (\beta + 1)x - (1, 1)$.

Outline

1. Kari/Culik, Jeandel–Rao, Ammann sets of Wang tiles
2. Metallic mean Wang tiles
3. Stone inflations and 4-to-2 Rauzy fractals
4. **Open questions**

Open questions

- Describe Jeandel-Rao tiles as the instances of a **computer chip**.
- Which **other algebraic numbers** arise in aperiodic tilings ?
- Describe a **Tribonacci** set of Wang tiles
  and its 4-dimensional Rauzy fractal

  (a Tribonacci set of Wang tiles must exist following Mozes 1989)
- Find **geometric shapes** with Ammann bars on them associated with metallic-mean Wang tiles for $n > 2$.

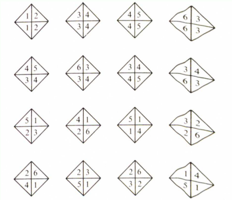


Figure 11.1.13
The 16 Wang tiles that correspond to the tiles of Figure 11.1.12.
These form the smallest known aperiodic set.

Figure 11.1.13

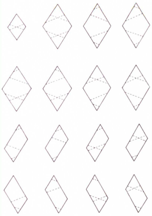


Figure 11.1.12
The 16 tiles that arise as indicated in Figure 11.1.11.

Figure 11.1.12

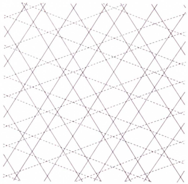


Figure 11.1.11
The Ammann bars of Figure 11.1.10 after the tiles have been detected. The solid bars are to be regarded as the edges of a new tiling by rhombs and parallelograms, the dashed bars are to be regarded as markings on the tiles specifying the matching condition.

Figure 11.1.11

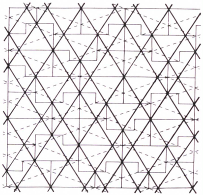


Figure 11.1.10
A tiling by the set A2 of Ammann prototiles with the four families of Ammann bars indicated, two by solid and two by dashed lines.

Figure 11.1.10

Branko Grünbaum and G. C. Shephard. 1987